Fractional Factorial Designs

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Fractional Factorial Designs

- In the context of two-level factors, a fractional factorial design is when ℓ factors are investigated in 2^k runs, where $\ell > k$.
- The full design would have 2^ℓ runs.
- If $\ell = k + 1$, this is a half-fraction, since 2^k is half of 2^ℓ.
- If $\ell = k + 2$, this is a quarter-fraction, and if $\ell = k + 4$, this is a sixteenth-fraction, and so on.

Confounding

Run	Temperature	Catalyst
1	250°	1.0%
2	200°	1.5%
3	250°	1.0%
4	200°	1.5%

The temperature effect and the catalyst effect are completely confounded. $(y_2 + y_4)/2 - (y_1 + y_3)/2$ estimates the change in yield from reducing the temperature from 250°to 200°, but the exact same number estimates the change in yield from increasing the catalyst percentage from 1.0% to 1.5%. We can't tell which is important, or what mixture of temperature effect and catalyst effect we are estimating. These effects are completely confounded.

Partial Confounding

Partial confounding happens when predictors are correlated with each other, but not 100% correlated. For example, suppose I try to predict the GPA of undergraduate BME students from x_1 = parents' educational attainment and x_2 = family income. Since these are likely correlated, part of the prediction comes from a common factor of the two predictors, and part from the unique parts of each predictor (suitably defined). The effects of the two predictors are partially confounded.

In designed experiments, we generally either have no confounding of effects, or predictable complete confounding of effects. In many designs, effects that are not completely confounded are orthogonal.

Complete confounding of A and B means that the calculated effect is either due to A or to B or to some mixture of the two.

Conceptual Experiment on Floor Wax

Factors to be studied were as follows:

Label	Definition	$-$	$+$
A	Catalyst (%)	1	1.5
B	Additive (%)	0.25	0.50
C	Emulsifier P (%)	2	3
D	Emulsifier Q (%)	1	2
E	Emulsifier R (%)	1	2

Ordinarily, this would take $2^5 = 32$ runs, but it was decided to run a quarter fraction in 8 runs. The first three factors were varied as a full factorial in **A**, **B**, **C**, while **D** was run according to the **BC** interaction and **E** was run according to the **ABC** interaction.

With relations $\mathbf{I} = \mathbf{BCD}$ and $\mathbf{I} = \mathbf{ABCE}$, (and $\mathbf{I} = \mathbf{ADE}$), the confounding table looked like this:

A	**ABCD**	**BCE**	**DE**
B	**CD**	**ACE**	**ABDE**
C	**BD**	**ABE**	**ACDE**
D	**BC**	**ABCDE**	**AE**
E	**BCDE**	**ABC**	**AD**

Each main effect is confounded with one or two two-way interactions, but no two main effects are confounded.

The important part of the confounding table is the lowest order of the confounding relations, which in this case is confounding of main effects and two-way interactions.

A	**DE**	
B	**CD**	
C	**BD**	
D	**BC**	**AE**
E	**AD**	

Each main effect is confounded with one or two two-way interactions, but no two main effects are confounded. Some two-way interactions are not confounded with main effects (**AB/CE, AC/BE)**.

There were six responses, as follows:

y_1	Hazy Y/N?
y_2	Adheres Y/N?
y_3	Grease on Top of Film Y/N?
y_4	Grease Under Film Y/N?
y_5	Dull, Adjusted pH Y/N?
y_6	Dull, Original pH Y/N?

These are all qualitative, but that was suitable for the application.

D = **BC**, **E** = **ACE**

```
> wax <- read.table("tab0601.dat",header=T)
> wax
  run  A  B  C  D  E  y1  y2  y3  y4       y5       y6
1   1 -1 -1 -1  1 -1  no  no yes  no slightly      yes
2   2  1 -1 -1  1  1  no yes yes yes slightly      yes
3   3 -1  1 -1 -1  1  no  no  no yes       no       no
4   4  1  1 -1 -1 -1  no yes  no  no       no       no
5   5 -1 -1  1 -1  1 yes  no  no yes       no slightly
6   6  1 -1  1 -1 -1 yes yes  no  no       no       no
7   7 -1  1  1  1 -1 yes  no yes  no slightly      yes
8   8  1  1  1  1  1 yes yes yes yes slightly      yes
```

Haze (y_1) correlates perfectly with **C**, Emulsifier P. (Other coefficients are $\therefore$ 0).
Adherence (y_2) correlates perfectly with **A**, Catalyst. (Other coefficients are 0).
Grease on top of film (y_3) correlates perfectly with **D**, Emulsifier Q. (Other coefficients are 0).
Grease under film (y_4) correlates perfectly with **E**, Emulsifier R. (Other coefficients are 0).
Dull, adjusted pH (y_5) correlates perfectly with **D**, Emulsifier Q. (Other coefficients are 0).
Dull, original pH (y_5) correlates almost perfectly with **D**, Emulsifier Q.

Set factor **A** to + (1.5% catalyst).
Factor **B** (additive %) seems inactive.
Set factor **C** to − (2% emulsifier P).
Set factor **D** to − (1% emulsifier Q).
Set factor **E** to − (1% emulsifier R).

Stability of a New Product

Section 6.2 of the text concerns an experiment with four factors in eight runs on the stability R of a product, with a desired level of 25 on the scale used. The factors varied were as follows (book has $-/+$ reversed):

Label	Definition	$-$	$+$
A	acid concentration	20	30
B	catalyst concentration	1	2
C	temperature	100	150
D	monomer concentration	25	50

The design was a full factorial in **A, B**, and **C** with **D** identified with the **abc** interaction in randomized order.

With defining relation $\mathbf{I} = \mathbf{ABCD}$, main effects are not confounded with two-way interactions. The confounding pattern of two-way interactions is as follows

AB	**CD**
AC	**BD**
AD	**BC**

We can estimate the intercept, the four main effects, and the three confounded two-way interactions, but with no error term. Or we can estimate the main effects model with 3df for error.

```
> product <- read.table("tab0602.dat",header=T)
> product
  test  A  B  C  D  y
1    1 -1 -1 -1 -1 20
2    2  1 -1 -1  1 14
3    3 -1  1 -1  1 17
4    4  1  1 -1 -1 10
5    5 -1 -1  1  1 19
6    6  1 -1  1 -1 13
7    7 -1  1  1 -1 14
8    8  1  1  1  1 10
```

The goal was to find a formulation that had a stability rating of $R = 25$. The best observation in the experiment was the first observation with all factors at the $-$ level and that reading was at $R = 20$.

```
> product.lm1 <- lm(y~A*B+A*C+A*D,data=product)
> product.lm2 <- lm(y~A+B+C+D,data=product)
> summary(product.lm1)
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   14.625        NaN     NaN      NaN
A             -2.875        NaN     NaN      NaN
B             -1.875        NaN     NaN      NaN
C             -0.625        NaN     NaN      NaN
D              0.375        NaN     NaN      NaN
A:B            0.125        NaN     NaN      NaN
A:C            0.375        NaN     NaN      NaN
A:D           -0.125        NaN     NaN      NaN
> summary(product.lm2)
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  14.6250     0.2394  61.101 9.66e-06 ***
A            -2.8750     0.2394 -12.011  0.00124 **
B            -1.8750     0.2394  -7.833  0.00433 **
C            -0.6250     0.2394  -2.611  0.07960 .
D             0.3750     0.2394   1.567  0.21517

Residual standard error: 0.677 on 3 degrees of freedom
Multiple R-squared:  0.9862,    Adjusted R-squared:  0.9679
F-statistic: 53.73 on 4 and 3 DF,  p-value: 0.004005
```

The interaction effects are relatively small, and only the **A** and**B** effects look large. None of the predicted values at the design points exceed 20. But the coefficients for the two large effects point towards possible formulations that might have higher stability. The gradient vector, normalized to have coefficient -1 for the **A** effect is $-\mathbf{A} - 0.65\mathbf{B}$

Some point that lie along the gradient, with the original units, and the predicted values are given below.

A Coded	Acid Conc.	B Coded	Catalyst Conc.	Prediction
-1	20	-0.65	1.175	18.7
-2	15	-1.30	0.85	22.8
-3	10	-1.95	0.525	26.9

A few actual runs in this direction were able to obtain a product for the first time with a stability R greater than 25.

Half-Fraction in Roller Bearing Modification

- Section 6.3 has this example from Hellstrand (1989) (linked on the web page).
- AB SKF (Swedish: Svenska Kullagerfabriken; 'Swedish Ball Bearing Factory') is a Swedish bearing and seal manufacturing company founded in Gothenburg, Sweden, in 1907.
- SKF is the world's largest bearing manufacturer, and employs 44,000 people in 108 manufacturing units. SKF is one of the largest companies in Sweden and among the largest public companies in the world. [Wikipedia]

Quality and Productivity

- Statistical process control and quality improvement by experimental design was definitely trending in 1989.
- BHH states that SKF (not named) lost a contract for roller bearings in washing machines that would be robust to ill-balanced loads.
- Hellstrom reports a 2^3 factorial design using
  - Inner ring heat treatment,
  - Outer ring osculation (ratio between the ball diameter and the radius of the outer ring raceway, and
  - Cage design.

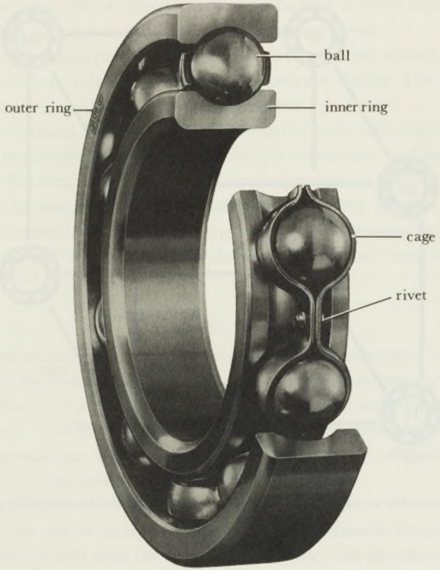


FIGURE 2. Standard deep groove ball bearing.

```
> bearings1
   A  B  C   y
1 -1 -1 -1  17
2  1 -1 -1  26
3 -1  1 -1  25
4  1  1 -1  85
5 -1 -1  1  19
6  1 -1  1  16
7 -1  1  1  21
8  1  1  1 128
> summary(lm(y~A*B*C,
  data=bearings1))
Coefficients:
            Estimate
(Intercept)   42.125
A             21.625
B             22.625
C              3.875
A:B           20.125
A:C            4.375
B:C            5.875
A:B:C          7.375
```

These are the data reported in the printed paper Hellstrand (1989). The data in BHH are different and likely were obtained either from the original presentation to the Royal Society of London, from a more complete and longer draft, or privately (Box in particular was heavily involved in industrial consulting).

The coding is -1 is standard and $+1$ is modified. The response y is lifetime on test in hours.

Factors A (inner ring heat treatment) and B (outer ring osculation) look important, as does the A:B interaction. Factor C (cage design) appears inert. Both of these were previously unknown and resulted in a much better product.

BHH Roller Bearings Example

Section 6.3 concerns an experiment with four factors in eight runs on the failure rate y (reciprocal of time to failure) of roller bearings. The factors varied were as follows:

Label	Definition	$-$	$+$
A	ball mfg	std	mod
B	cage design	std	mod
C	type of grease	std	mod
D	amount of grease	std	mod

The design was a full factorial in **A, B**, and **C** with **D** identified with the **abc** interaction in randomized order. Main effects are not confounded with two-way interactions, but two-way interactions are confounded.

```
> bearings2
   A  B  C  D  y
1 -1 -1 -1 -1 16
2  1 -1 -1  1  7
3 -1  1 -1  1 14
4  1  1 -1 -1  5
5 -1 -1  1  1 11
6  1 -1  1 -1  7
7 -1  1  1 -1 13
8  1  1  1  1  4
> summary(lm(y~A*B+A*C+A*D,
  data=bearings2))
Coefficients:
            Estimate
(Intercept)    9.625
A             -3.875
B             -0.625
C             -0.875
D             -0.625
A:B           -0.625
A:C            0.625
A:D            0.375
```

The effects for B and D are the same size as two of the three interaction terms, so at most A and C have been shown to be active.

If we fit a model only with A, C, and the interaction, only A looks important, so that should probably be the conclusion, with manufacturing condition of the balls important, and (contra BHH), none of the other factors including cage design.

Anatomy of a Half-Fraction

Suppose we have four factors, **A, B, C**, and **D**, and we run a half fraction of the 2^4 design by identifying factor **D** with the **ABC** interaction. With defining relation $\mathbf{I} = \mathbf{ABCD}$, main effects are not confounded with two-way interactions. The confounding pattern of all the effects is as follows

A	**BCD**
B	**ACD**
C	**ABD**
D	**ABC**
AB	**CD**
AC	**BD**
AD	**BC**
I	**ABCD**

Anatomy of a Half-Fraction

A	BCD
B	ACD
C	ABD
D	ABC
AB	CD
AC	BD
AD	BC
I	ABCD

The coefficient or effect calculated (as $1/4$ of the dot product of a $-1/+1$ column) is the sum of the two theoretical effects. Each is an *alias* of the other with which it is *confounded*. (The $1/4$ is because it is the difference of two averages of 4. The `lm()` coefficient is half that since it is per unit change.)

Design Resolution

The *resolution* of a design is the length of the shortest word which is confounded with the identity. With a half fraction, there is only one word confounded with the identity and its length is the resolution.

If we have a 2^{4-1} design in which $\mathbf{D}$ is confounded with $\mathbf{ABC}$, then $\mathbf{I} = \mathbf{ABCD}$ and the design is of resolution 4. If for some reason we wanted to confound $\mathbf{D}$ with $\mathbf{BC}$ then $\mathbf{I} = \mathbf{BCD}$ and the design is of resolution 3.

Design Resolution

If we have a 2^{5-2} design and we confound **D** with **AB** and **E** with **BC**, then **DE** is confounded with **AC** and the defining relations are $\mathbf{I} = \mathbf{ABD} = \mathbf{BCE} = \mathbf{ACDE}$ and the design is of resolution 3.

We denote the 2^{4-1} on the last slide by 2^{4-1}_{IV}, with resolution in Roman numerals, and the design above by 2^{5-2}_{III}. In general, a 2^{p-q}_{r} design is of size 2^{p-q} runs, with p variables and of resolution (Roman) r.

Higher Order Interactions

A main effect is a difference quotient-estimate of the partial derivative of the response function. Higher-order effects are difference quotient estimates of mixed partial derivatives. In general, for smooth functions, we would presume that for small regions not at the maximum or minimum a linear approximation is satisfactory. For slightly larger regions, a quadratic model might suffice, and then a cubic, and so on. In general, we might think that the larger the order, the smaller the coefficient. Thus, it is often the case that higher order interactions may be negligible compared to the noise at some point.

Redundancy

If we have a full 2^5 factorial in five variables with 32 runs, then we can estimate an intercept, 5 main effects, 10 two-factor interactions, 10 three-factor interactions, 5 four-factor interactions, and 1 five-factor interaction.

If three, four, and five factor interactions are negligible compared to the noise, then we have 16df for error, which might be more than needed.

Thus we might be inclined to add a sixth factor, and get a 2_{VI}^{6-1} or even a seventh factor to obtain a 2_{IV}^{7-2} and get greater efficiency.

Parsimony

The *Pareto Principle* is that of the important few and the trivial many. If we conceive of 10 factors that may influence the strength of engineered cartilage, and run an experiment varying all 10, most likely only a few will be important, at least over the range that we vary them. Too large a range leads to nonlinearity, while too short a range leads to no measurable effect, and we often, when starting out, don't know the correct range. That can be fixed in follow-up experiments.

Projectivity

If we have a 2^{3-1}_{III} in which $\mathbf{C} = \mathbf{AB}$, then if any of the factors is inert, the design becomes a full 2^2 factorial in the remaining elements. The design is of *projectivity* $P = 2$.

a	b	ab
A	B	C
-1	-1	$+1$
$+1$	-1	-1
-1	$+1$	-1
$+1$	$+1$	$+1$

$\mathbf{C} = \mathbf{AB}$, $\mathbf{A} = \mathbf{BC}$, and $\mathbf{B} = \mathbf{AC}$ because $\mathbf{I} = \mathbf{ABC}$.

Projectivity

The 2^{4-1}_{IV} in which $\mathbf{D} = \mathbf{ABC}$, is of projectivity $P = 3$, because if any one of the four factors is inert, the remaining three form a full 2^3 factorial.

This can be checked by multiplying the columns, or noting that we have seven contrast columns originally labeled **A, B, C, AB, AC, BC, ABC**. If **A** is inert, we have already columns for **B, C, D, BC**.

The three remaining original columns **A, AB, AC** are repurposed as $\mathbf{BD} = \mathbf{AC}$, $\mathbf{CD} = \mathbf{AB}$, and $\mathbf{BCD} = \mathbf{A}$, and so the seven original columns are now the full factorial in **B, C, D**. The same calculations work if **B** or **C** is inert.

Seven factors in Eight Runs

Section 6.5 in the text presents an experiment in bicycle hill climbing, with seven factors being varied:

Factor	Value	Definition	-1	$+1$
A	**a**	seat	up	down
B	**b**	dynamo	off	on
C	**c**	handlebars	up	down
D	**ab**	gear	low	medium
E	**ac**	raincoat	on	off
F	**bc**	breakfast	yes	no
G	**abc**	tires	hard	soft

The response is time in seconds to climb the hill. Preliminary test runs suggests that the standard deviation of replicate runs is about 3 seconds. An "effect" is the difference between the average of four $+1$ runs and the average of four -1 runs. The standard deviation of an effect is then about $\sqrt{3^2/4 + 3^2/4} = 2.1$. Since the regression coefficient in the $-1/+1$ coding is half the "effect'," the standard deviation is about 1.05.

```
> bike2
  run seat dynamo handlebars gear raincoat breakfast tires  y
1   1   -1     -1         -1    1        1         1    -1 69
2   2    1     -1         -1   -1       -1         1     1 52
3   3   -1      1         -1   -1        1        -1     1 60
4   4    1      1         -1    1       -1        -1    -1 83
5   5   -1     -1          1    1       -1        -1     1 71
6   6    1     -1          1   -1        1        -1    -1 50
7   7   -1      1          1   -1       -1         1    -1 59
8   8    1      1          1    1        1         1     1 88
> summary(lm(y~seat+dynamo+handlebars+gear+raincoat+breakfast+tires,data=bike2))
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)    66.50        NaN     NaN      NaN
seat            1.75        NaN     NaN      NaN
dynamo          6.00        NaN     NaN      NaN    <= probably real
handlebars      0.50        NaN     NaN      NaN
gear           11.25        NaN     NaN      NaN    <= probably real
raincoat        0.25        NaN     NaN      NaN
breakfast       0.50        NaN     NaN      NaN
tires           1.25        NaN     NaN      NaN
```

Confounding Pattern

The defining relations are generated by $\mathbf{D} = \mathbf{AB}$, $\mathbf{E} = \mathbf{AC}$, $\mathbf{F} = \mathbf{BC}$, and $\mathbf{G} = \mathbf{ABC}$, or

$$\mathbf{I} = \mathbf{ABD} = \mathbf{ACE} = \mathbf{BCF} = \mathbf{ABCG}$$

The remaining elements of the defining relations are the products of these two at a time:

$$\mathbf{I} = \mathbf{BCDE} = \mathbf{ACDF} = \mathbf{CDG} = \mathbf{ABEF} = \mathbf{BEG} = \mathbf{AFG}$$

and the products three at a time or all four:

$$\mathbf{I} = \mathbf{DEF} = \mathbf{ADEG} = \mathbf{BDFG} = \mathbf{CEFG} = \mathbf{ABCDEFG}$$

The smallest word in the defining relation is of length 3, so this is a resolution 3 design (2^{7-4}_{III}).

Ignoring interactions of more than two factors, the confounding pattern is as follows:

Variable	Result		Confounding
Seat	**A** =	1.75	**A** + **BD** + **CE** + **FG**
Dynamo	**B** =	6.00	**B** + **AD** + **CF** + **EG**
Handlebars	**C** =	0.50	**C** + **AE** + **BF** + **DG**
Gear	**D** =	11.25	**D** + **AB** + **CG** + **EF**
Raincoat	**E** =	0.25	**E** + **AC** + **BG** + **DF**
Breakfast	**F** =	0.50	**F** + **BC** + **AG** + **DE**
Tires	**G** =	1.25	**G** + **CD** + **BE** + **AF**

Note that the third largest coefficient **A** is also the **BD** interaction of the two largest coefficients. Also note that each numerical coefficient is the *sum* of the four listed effects.

Projection

- Any two factors generate a replicated 2^2 design. Some, but not all, triples generate a full 2^3 design, but some just generate a 2^2.
- For example, the three effects **A, B, D** only generate a 2^2 because $\mathbf{AB} = \mathbf{D}$, $\mathbf{AD} = \mathbf{B}$, and $\mathbf{BD} = \mathbf{A}$. Thus $P = 2$.
- Of the 35 possible triples of columns, 28 generate the full 2^3 factorial.

Nodal Designs

A *nodal* design is one that for the given number of runs has the largest number of factors at a given resolution. For example, with 8 runs, resolution 4 designs have all words in the defining relation at least of length 4, which means there have to be at least 4 factors. The standard 2^{4-1}_{IV} in which $\mathbf{I} = \mathbf{ABCD}$ has 8 runs. This design is nodal.

With 5 factors in 8 runs, resolution 4 is not possible. To get down from 32 runs to 8 and satisfy resolution 4 requires two generators of the defining relation, each of length 4 or greater, whose product is also of length 4 or greater. But any two distinct words from 5 letters that are of length 4 must overlap in exactly 3, so that their product is of length 2. For example, if $\mathbf{I} = \mathbf{ABCD} = \mathbf{BCDE}$, then $\mathbf{I} = \mathbf{AE}$, and the design is of resolution 2. Also $\mathbf{I} = \mathbf{ABCDE}$ along with a relation of length 4 generates a relation of length 1.

Nodal Designs

With 8 runs, a resolution 3 design needs to have defining relations with no words shorter than 3 letters. The standard 2^{7-4}_{III} has no words in the defining relation shorter than 3 letters (see slide 31). Clearly, 7 factors is the largest that can be accommodated, so this design is nodal.

All other designs with 8 runs can be accommodated by dropping one or more variables that are identified with the interactions of the first three columns. If there are 5–7 factors, the resolution cannot be better than 3.

Derived Design for a Laboratory Experiment

- A reaction needs to be optimized to increase the percentage yield.
- The current conditions are thought to be far from the optimum, so probably main effects are most important to find "uphill."
- There were five variables that might be changed to move towards increased yield.
- But one or more might be inert at this design area.
- And after some thought, one interaction might be important.
- A fractional factorial seems appropriate

Factor	Definition	-1	$+1$	Units
1	concentration of γ	94	96	%
2	proportion of γ to α	3.85	4.15	mol/mol
3	amount of solvent	280	310	cm^3
4	proportion of β to α	3.5	5.5	mol/mol
5	reaction time	2	4	hr

We can use factors **1–3** as the first three columns $\mathbf{1} = \mathbf{A} = \mathbf{a}, \mathbf{2} = \mathbf{B} = \mathbf{b}, \mathbf{3} = \mathbf{C} = \mathbf{c}$ in the standard 8 run, 7 factor design. The **13** interaction might be important, so we do not use the **ac** column. If we use the **ab** column for **4=D=ab** and the **abc** column for **5=G=abc**, then the only other confounding is $\mathbf{I} = \mathbf{CDG}$ which is also free of the $\mathbf{13} = \mathbf{AC}$ interaction.

```
> yield
  run  A  B  C  D  G    y
1   1 -1 -1 -1  1 -1 77.1
2   2  1 -1 -1 -1  1 68.9
3   3 -1  1 -1 -1  1 75.5
4   4  1  1 -1  1 -1 72.5
5   5 -1 -1  1  1  1 67.9
6   6  1 -1  1 -1 -1 68.5
7   7 -1  1  1 -1 -1 71.5
8   8  1  1  1  1  1 63.7
> summary(lm(y~A+B+C+D+G+A:C,data=yield))
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)     70.7        0.4  176.75   0.0036 **
A               -2.3        0.4   -5.75   0.1096
B                0.1        0.4    0.25   0.8440
C               -2.8        0.4   -7.00   0.0903 .
D               -0.4        0.4   -1.00   0.5000
G               -1.7        0.4   -4.25   0.1471
A:C              0.5        0.4    1.25   0.4296
```

Effects from **A(1), C(3), G(5)** are all large and negative, so reducing concentration of γ, amount of solvent, and reaction time should improve yield. Moving in these directions gave an eventual yield of 84%, compared to the current best of 77%.

Sequential Experimentation

- Every experiment should do something; no experiment can do everything.
- If an experiment suggests that the best settings are not in the present region, then we can move, perhaps along the gradient, and set up another design. The full use of this is in response surface methods (Chapter 12 and Box and Draper).
- If the present experiment leaves important unanswered questions in the current region, then the design can be augmented by addition runs. We will consider the use of foldover design augmentation.

Foldover Design

- Given a design matrix, a foldover is a design with all the signs reversed.
- By itself, this generates the same amount of information as the original design.
- But together, this can lead to more complete information.
- Suppose we have four runs with three factors (2^{3-1}_{III}), where **C** is identified with the **AB** interaction.

Original

a	b	ab
A	**B**	**C**
$-$	$-$	$+$
$+$	$-$	$-$
$-$	$+$	$-$
$+$	$+$	$+$

Foldover

a	b	-ab
A	**B**	**C**
$+$	$+$	$-$
$-$	$+$	$+$
$+$	$-$	$+$
$-$	$-$	$-$

Combined

ac	bc	abc	c
A	**B**	**C**	**W**
$+$	$+$	$-$	$-$
$-$	$+$	$+$	$-$
$+$	$-$	$+$	$-$
$-$	$-$	$-$	$-$
$-$	$-$	$+$	$+$
$+$	$-$	$-$	$+$
$-$	$+$	$-$	$+$
$+$	$+$	$+$	$+$

The original design can estimate the intercept and three main effects, but has no df for error. The combined design has the foldover on top so the pattern of signs matches some columns of the full factorial. The **W** factor is the block in which the runs were located. Although the patterns of $-/+$ are different from the first three columns of the standard table, they contain all 8 possibilities and could be reordered into the standard order. This is now a 2^{4-1}_{IV} design that allows estimation of all the main effects and two-factor interactions of **A**, **B**, **C** as well as the block effect (but no error df). Alternatively, the main effects and blocks can be estimated with 3df for error.

We can start with the smaller design and see if we need to augment it and in which way.

Single Column Foldover

Section 6.5 in the text presents an experiment in bicycle hill climbing, with seven factors being varied in eight runs:

Factor	Value	Definition	-1	$+1$
A	**a**	seat	up	down
B	**b**	dynamo	off	on
C	**c**	handlebars	up	down
D	**ab**	gear	low	medium
E	**ac**	raincoat	on	off
F	**bc**	breakfast	yes	no
G	**abc**	tires	hard	soft

```
> bike2
  run seat dynamo handlebars gear raincoat breakfast tires  y
1   1   -1     -1         -1    1        1         1    -1 69
2   2    1     -1         -1   -1       -1         1     1 52
3   3   -1      1         -1   -1        1        -1     1 60
4   4    1      1         -1    1       -1        -1    -1 83
5   5   -1     -1          1    1       -1        -1     1 71
6   6    1     -1          1   -1        1        -1    -1 50
7   7   -1      1          1   -1       -1         1    -1 59
8   8    1      1          1    1        1         1     1 88
> summary(lm(y~seat+dynamo+handlebars+gear+raincoat+breakfast+tires,data=bike2))
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)    66.50        NaN     NaN      NaN
seat            1.75        NaN     NaN      NaN
dynamo          6.00        NaN     NaN      NaN    <= probably real
handlebars      0.50        NaN     NaN      NaN
gear           11.25        NaN     NaN      NaN    <= probably real
raincoat        0.25        NaN     NaN      NaN
breakfast       0.50        NaN     NaN      NaN
tires           1.25        NaN     NaN      NaN
```

Factors **B** (dynamo) and **D** (gear) are dominant. The cyclist/experimenter had previously thought that the gear change might interact with other variables. For example, with a higher gear, the cyclist might stand, making seat height irrelevant. He reran the previous eight runs except with the signs of **D** reversed.

Confounding Pattern

The defining relations for the first half are

$$\mathbf{I} = \mathbf{ABD} = \mathbf{ACE} = \mathbf{BCF} = \mathbf{ABCG}$$
$$\mathbf{I} = \mathbf{BCDE} = \mathbf{ACDF} = \mathbf{CDG} = \mathbf{ABEF} = \mathbf{BEG} = \mathbf{AFG}$$
$$\mathbf{I} = \mathbf{DEF} = \mathbf{ADEG} = \mathbf{BDFG} = \mathbf{CEFG} = \mathbf{ABCDEFG}$$

And for the new runs

$$\mathbf{I} = -\mathbf{ABD} = \mathbf{ACE} = \mathbf{BCF} = \mathbf{ABCG}$$
$$\mathbf{I} = -\mathbf{BCDE} = -\mathbf{ACDF} = -\mathbf{CDG} = \mathbf{ABEF} = \mathbf{BEG} = \mathbf{AFG}$$
$$\mathbf{I} = -\mathbf{DEF} = -\mathbf{ADEG} = -\mathbf{BDFG} = \mathbf{CEFG} = -\mathbf{ABCDEFG}$$

and so overall,

$$\mathbf{I} = \mathbf{ACE} = \mathbf{BCF} = \mathbf{ABCG}$$
$$\mathbf{I} = \mathbf{ABEF} = \mathbf{BEG} = \mathbf{AFG}$$
$$\mathbf{I} = \mathbf{CEFG}$$

Which shows that **D** and its two-factor interactions are not confounded with anything. If we add a block factor, all the words that are + in one block and − in the other are confounded with blocks.

Ignoring interactions of more than two factors, the confounding pattern in the two blocks is as follows:

Variable	Result		Confounding
Seat	$\mathbf{A} =$	1.75	$\mathbf{A} + \mathbf{BD} + \mathbf{CE} + \mathbf{FG}$
Dynamo	$\mathbf{B} =$	6.00	$\mathbf{B} + \mathbf{AD} + \mathbf{CF} + \mathbf{EG}$
Handlebars	$\mathbf{C} =$	0.50	$\mathbf{C} + \mathbf{AE} + \mathbf{BF} + \mathbf{DG}$
Gear	$\mathbf{D} =$	11.25	$\mathbf{D} + \mathbf{AB} + \mathbf{CG} + \mathbf{EF}$
Raincoat	$\mathbf{E} =$	0.25	$\mathbf{E} + \mathbf{AC} + \mathbf{BG} + \mathbf{DF}$
Breakfast	$\mathbf{F} =$	0.50	$\mathbf{F} + \mathbf{BC} + \mathbf{AG} + \mathbf{DE}$
Tires	$\mathbf{G} =$	1.25	$\mathbf{G} + \mathbf{CD} + \mathbf{BE} + \mathbf{AF}$

Variable	Result		Confounding
Seat	$\mathbf{A} =$	0.375	$\mathbf{A} - \mathbf{BD} + \mathbf{CE} + \mathbf{FG}$
Dynamo	$\mathbf{B} =$	5.125	$\mathbf{B} - \mathbf{AD} + \mathbf{CF} + \mathbf{EG}$
Handlebars	$\mathbf{C} =$	1.375	$\mathbf{C} + \mathbf{AE} + \mathbf{BF} - \mathbf{DG}$
Gear	$\mathbf{D} =$	12.625	$\mathbf{D} - \mathbf{AB} - \mathbf{CG} - \mathbf{EF}$
Raincoat	$\mathbf{E} =$	-0.875	$\mathbf{E} + \mathbf{AC} + \mathbf{BG} - \mathbf{DF}$
Breakfast	$\mathbf{F} =$	-1.125	$\mathbf{F} + \mathbf{BC} + \mathbf{AG} - \mathbf{DE}$
Tires	$\mathbf{G} =$	-0.375	$\mathbf{G} - \mathbf{CD} + \mathbf{BE} + \mathbf{AF}$

B and **D** look large in both fractions, and nothing else much does. Note how the interactions with **D** cancel out in the two halves.

Here is the confounding pattern (up to two-factor interactions) of the whole design, including the blocks effect.

Variable		Confounding
Seat	$\mathbf{A} =$	$\mathbf{A} + \mathbf{CE} + \mathbf{FG}$
Dynamo	$\mathbf{B} =$	$\mathbf{B} + \mathbf{CF} + \mathbf{EG}$
Handlebars	$\mathbf{C} =$	$\mathbf{C} + \mathbf{AE} + \mathbf{BF}$
Gear	$\mathbf{D} =$	$\mathbf{D}$
Raincoat	$\mathbf{E} =$	$\mathbf{E} + \mathbf{AC} + \mathbf{BG}$
Breakfast	$\mathbf{F} =$	$\mathbf{F} + \mathbf{BC} + \mathbf{AG}$
Tires	$\mathbf{G} =$	$\mathbf{G} + \mathbf{BE} + \mathbf{AF}$
Blocks	$\mathbf{H} =$	$\mathbf{H}$

Both **D** and the blocks effect are estimated free of two-factor interactions.

```
> summary(lm(y~D*(A+B+C+E+F+G)+H,data=bike3))
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  67.3125     0.6875  97.909   0.0065 **
D            11.9375     0.6875  17.364   0.0366 *
A             1.0625     0.6875   1.545   0.3656
B             5.5625     0.6875   8.091   0.0783 .
C             0.9375     0.6875   1.364   0.4028
E            -0.3125     0.6875  -0.455   0.7284
F            -0.3125     0.6875  -0.455   0.7284
G             0.4375     0.6875   0.636   0.6392
H             0.8125     0.6875   1.182   0.4471
D:A           0.4375     0.6875   0.636   0.6392
D:B           0.6875     0.6875   1.000   0.5000
D:C           0.8125     0.6875   1.182   0.4471
D:E           0.8125     0.6875   1.182   0.4471
D:F           0.5625     0.6875   0.818   0.5635
D:G          -0.4375     0.6875  -0.636   0.6392
```

```
> summary(lm(y~A+B+C+D+E+F+G+H,data=bike3))
Residuals:
    Min      1Q  Median      3Q     Max 
-3.5625 -0.8750 -0.3125  1.5625  2.4375 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  67.3125     0.6508 103.429 2.08e-12 ***
A             1.0625     0.6508   1.633    0.147    
B             5.5625     0.6508   8.547 5.96e-05 ***
C             0.9375     0.6508   1.441    0.193    
D            11.9375     0.6508  18.343 3.55e-07 ***
E            -0.3125     0.6508  -0.480    0.646    
F            -0.3125     0.6508  -0.480    0.646    
G             0.4375     0.6508   0.672    0.523    
H             0.8125     0.6508   1.248    0.252    
---
Residual standard error: 2.603 on 7 degrees of freedom
Multiple R-squared:  0.9835,    Adjusted R-squared:  0.9646 
F-statistic: 52.09 on 8 and 7 DF,  p-value: 1.495e-05
```

Only the **B** and **D** effects seem large, and none of the interactions with **D** or the block effect are large. In this case, the conclusions are unchanged from the original fraction.

Multicolumn Foldover

- A number of similar chemical plants in different locations had been operating successfully for several years.
- In older plants, a particular filtration cycle took about 40 minutes, but in a newly constructed plant it took twice as long.
- Considering differences between the older and the newer plants, and the details of the filtration operation, seven factors were identified.
- There was little information and many opinions about which of these were most important in solving the problem.

It was thought that probably at most two factors of the seven were important, so the nodal seven-factor, eight-run design 2^{7-4}_{III} was used. The $-$ levels were apparently the current ones and the $+$ levels were possible changes.

	Factors	$-$	$+$
A	water supply	Town reservoir	Well
B	raw material	On site	Other
C	temperature	Low	High
D	recycle	Yes	No
E	caustic soda	Fast	Slow
F	filter cloth	New	Old
G	holdup time	Low	High

```
> filter
  run  A  B  C  D  E  F  G    y
1   1 -1 -1 -1  1  1  1 -1 68.4
2   2  1 -1 -1 -1 -1  1  1 77.7
3   3 -1  1 -1 -1  1 -1  1 66.4
4   4  1  1 -1  1 -1 -1 -1 81.0
5   5 -1 -1  1  1 -1 -1  1 78.6
6   6  1 -1  1 -1  1 -1 -1 41.2
7   7 -1  1  1 -1 -1  1 -1 68.7
8   8  1  1  1  1  1  1  1 38.7
> summary(lm(y~A+B+C+D+E+F+G,data=filter))
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  65.0875        NaN     NaN      NaN
A            -5.4375        NaN     NaN      NaN
B            -1.3875        NaN     NaN      NaN
C            -8.2875        NaN     NaN      NaN
D             1.5875        NaN     NaN      NaN
E           -11.4125        NaN     NaN      NaN
F            -1.7125        NaN     NaN      NaN
G             0.2625        NaN     NaN      NaN
```

The two runs with the lowest times were the only ones that had **A**, **C**, and **E** simultaneously changed and those factors had the largest negative coefficients.

Ignoring interactions of more than two factors, the confounding pattern is as follows:

Variable		Result	Confounding
water supply	$\mathbf{A} =$	-5.4375	$\mathbf{A} + \mathbf{BD} + \mathbf{CE} + \mathbf{FG}$
raw material	$\mathbf{B} =$	-1.3875	$\mathbf{B} + \mathbf{AD} + \mathbf{CF} + \mathbf{EG}$
temperature	$\mathbf{C} =$	-8.2875	$\mathbf{C} + \mathbf{AE} + \mathbf{BF} + \mathbf{DG}$
recyle	$\mathbf{D} =$	1.5875	$\mathbf{D} + \mathbf{AB} + \mathbf{CG} + \mathbf{EF}$
caustic soda	$\mathbf{E} =$	-11.4125	$\mathbf{E} + \mathbf{AC} + \mathbf{BG} + \mathbf{DF}$
filter cloth	$\mathbf{F} =$	-1.7125	$\mathbf{F} + \mathbf{BC} + \mathbf{AG} + \mathbf{DE}$
holdup time	$\mathbf{G} =$	0.2625	$\mathbf{G} + \mathbf{CD} + \mathbf{BE} + \mathbf{AF}$

A could be large, but also it could be **CE**.
C could be large, but also it could be **AE**.
E could be large, but also it could be **AC**.

So it could be **A**, **C**, **E**, or **A**, **C**, **AC**, or **A**, **E**, **AE**, or **C**, **E**, **CE**.

We could change all three of the factors and fix the problem for the moment, but without understanding why. Further study is needed to solve the problem. We will run a full foldover fraction of an additional eight runs.

Confounding Pattern

The defining relations for the first half are

$$\mathbf{I} = \mathbf{ABD} = \mathbf{ACE} = \mathbf{BCF} = \mathbf{ABCG}$$
$$\mathbf{I} = \mathbf{BCDE} = \mathbf{ACDF} = \mathbf{CDG} = \mathbf{ABEF} = \mathbf{BEG} = \mathbf{AFG}$$
$$\mathbf{I} = \mathbf{DEF} = \mathbf{ADEG} = \mathbf{BDFG} = \mathbf{CEFG} = \mathbf{ABCDEFG}$$

And for the new runs, the sign of each variable is changed, so signs of odd-length words are changed,

$$\mathbf{I} = -\mathbf{ABD} = -\mathbf{ACE} = -\mathbf{BCF} = \mathbf{ABCG}$$
$$\mathbf{I} = \mathbf{BCDE} = \mathbf{ACDF} = -\mathbf{CDG} = \mathbf{ABEF} = -\mathbf{BEG} = -\mathbf{AFG}$$
$$\mathbf{I} = -\mathbf{DEF} = \mathbf{ADEG} = \mathbf{BDFG} = \mathbf{CEFG} = -\mathbf{ABCDEFG}$$

and so overall,

$$\mathbf{I} = \mathbf{ABCG}$$
$$\mathbf{I} = \mathbf{BCDE} = \mathbf{ACDF} = \mathbf{ABEF}$$
$$\mathbf{I} = \mathbf{ADEG} = \mathbf{BDFG} = \mathbf{CEFG}$$

Which shows that $\mathbf{A}, \mathbf{C}, \mathbf{E}$ are not confounded with any two-factor interactions, and no two of $\mathbf{AC}, \mathbf{AE}, \mathbf{CE}$ are confounded with each other. The blocking factor $\mathbf{H}$ is confounded with any of the defining relations that change sign in the two halves, so mostly three-factor interactions.

```
> filter3
   run  A  B  C  D  E  F  G  H    y
1    1 -1 -1 -1  1  1  1 -1 -1 68.4
2    2  1 -1 -1 -1 -1  1  1 -1 77.7
3    3 -1  1 -1 -1  1 -1  1 -1 66.4
4    4  1  1 -1  1 -1 -1 -1 -1 81.0
5    5 -1 -1  1  1 -1 -1  1 -1 78.6
6    6  1 -1  1 -1  1 -1 -1 -1 41.2
7    7 -1  1  1 -1 -1  1 -1 -1 68.7
8    8  1  1  1  1  1  1  1 -1 38.7
-----------------------------------------
9    9  1  1  1 -1 -1 -1  1  1 66.7
10  10 -1  1  1  1  1 -1 -1  1 65.0
11  11  1 -1  1  1 -1  1 -1  1 86.4
12  12 -1 -1  1 -1  1  1  1  1 61.9
13  13  1  1 -1 -1  1  1 -1  1 47.8
14  14 -1  1 -1  1 -1  1  1  1 59.0
15  15  1 -1 -1  1  1 -1  1  1 42.6
16  16 -1 -1 -1 -1 -1 -1 -1  1 67.6
```

```
> summary(lm(y~A+B+C+D+E+F+G+H+A*B+A*C+A*D+A*E+B*C+B*D+C*D,data=filter3))
Coefficients:
             Estimate
(Intercept) 63.60625
A           -3.34375
B           -1.94375
C           -0.20625
D            1.35625
E           -9.60625
F           -0.03125
G           -2.15625
H           -1.48125
A:B          0.23125   AB + CG + EF
A:C         -1.80625   AC + BG + DF
A:D          0.55625   AD + CF + EG
A:E         -8.08125   AE + BF + DG
B:C         -1.68125   BC + AG + DE
B:D         -2.09375   BD + CE + FG
C:D          2.41875   CD + BE + AF
```

The largest (negative) coefficients are those of **A, E, AE**. The **C** coefficient is small. All of this means that water supply and caustic soda are the important factors (and not temperature).

- The foldover additional fraction augmented the eight-run 2_{III}^{7-4} design of projectivity 2 to a 16-run 2_{IV}^{7-3} design of projectivity 3.
- All the 16-run designs we have discussed have the same 16 columns, generated by the identity column, four dummy factors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$, their six two-factor interactions, their four three-factor interactions, and the four-factor interaction. $1 + 4 + 6 + 4 + 1 = 16$.
- These can be used to provide fractional factorials of resolution 3, 4, and 5.
- Four of these are nodal designs of the maximum number of factors for a given resolution.

This is the basic table of signs that generate the full 2^4 factorial.

run	a	b	c	d	ab	ac	ad	bc	bd	cd	abc	abd	acd	bcd	abcd
1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	-1	-1	1
2	1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1
3	-1	1	-1	-1	-1	1	1	-1	-1	1	1	1	-1	1	-1
4	1	1	-1	-1	1	-1	-1	-1	-1	1	-1	-1	1	1	1
5	-1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	1	-1
6	1	-1	1	-1	-1	1	-1	-1	1	-1	-1	1	-1	1	1
7	-1	1	1	-1	-1	-1	1	1	-1	-1	-1	1	1	-1	1
8	1	1	1	-1	1	1	-1	1	-1	-1	1	-1	-1	-1	-1
9	-1	-1	-1	1	1	1	-1	1	-1	-1	-1	1	1	1	-1
10	1	-1	-1	1	-1	-1	1	1	-1	-1	1	-1	-1	1	1
11	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1
12	1	1	-1	1	1	-1	1	-1	1	-1	-1	1	-1	-1	-1
13	-1	-1	1	1	1	-1	-1	-1	-1	1	1	1	-1	-1	1
14	1	-1	1	1	-1	1	1	-1	-1	1	-1	-1	1	-1	-1
15	-1	1	1	1	-1	-1	-1	1	1	1	-1	-1	-1	1	-1
16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Design	a	b	c	d	ab	ac	ad	bc	bd	cd	abc	abd	acd	bcd	abcd
Sixteen-Run Nodal Designs															
2^4	A	B	C	D											
2_{V}^{5-1}	A	B	C	D	—	—	—	—	—	—	—	—	—	—	P
2_{IV}^{8-4}	A	B	C	D	—	—	—	—	—	—	L	M	N	O	—
2_{III}^{15-11}	A	B	C	D	E	F	G	H	J	K	L	M	N	O	P

The last design has 15 factors in 16 runs and is resolution 3 because the shortest words are derived from identification of a main effect with a two-factor interaction like $\mathbf{I} = \mathbf{ABE}$. The second-to-last design has 8 factors in 16 runs. The initial words in the defining relation are

$$\mathbf{I} = \mathbf{ABCL} = \mathbf{ABDM} = \mathbf{ACDN} = \mathbf{BCDO}$$

and products of any two or three of these are also of length 4, while the product of all four is of length 8. Table 6.14c in the book shows the alias structure for the last three designs, and that of the 15-factor design can be used for any subset of factors between 9 and 14.

Nodal Half Replicate with Five Factors

The full 32-run design has

	Factor	$-$	$+$
A	feed rate (L/min)	10	15
B	catalyst (%)	1	2
C	agitation rate (rpm)	100	120
D	temperature (°C)	140	180
E	concentration	3	6

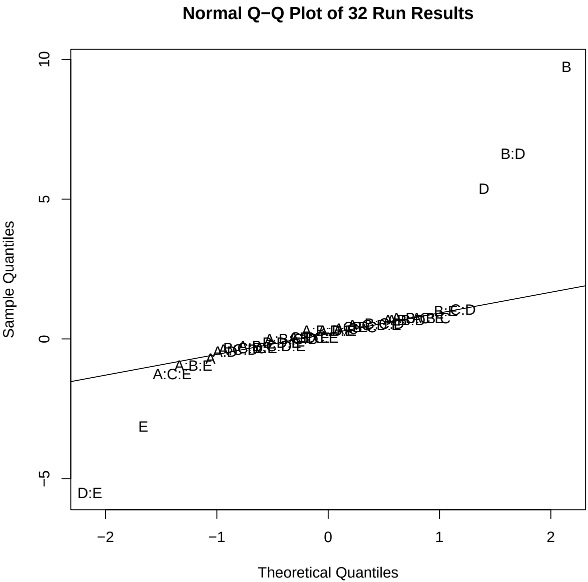
run	A	B	C	D	E	y
1	-1	-1	-1	-1	-1	61
2	1	-1	-1	-1	-1	53
3	-1	1	-1	-1	-1	63
4	1	1	-1	-1	-1	61
5	-1	-1	1	-1	-1	53
6	1	-1	1	-1	-1	56
7	-1	1	1	-1	-1	54
8	1	1	1	-1	-1	61
9	-1	-1	-1	1	-1	69
10	1	-1	-1	1	-1	61
11	-1	1	-1	1	-1	94
12	1	1	-1	1	-1	93
13	-1	-1	1	1	-1	66
14	1	-1	1	1	-1	60
15	-1	1	1	1	-1	95
16	1	1	1	1	-1	98
17	-1	-1	-1	-1	1	56
18	1	-1	-1	-1	1	63
19	-1	1	-1	-1	1	70
20	1	1	-1	-1	1	65
21	-1	-1	1	-1	1	59
22	1	-1	1	-1	1	55
23	-1	1	1	-1	1	67
24	1	1	1	-1	1	65
25	-1	-1	-1	1	1	44
26	1	-1	-1	1	1	45
27	-1	1	-1	1	1	78
28	1	1	-1	1	1	77
29	-1	-1	1	1	1	49
30	1	-1	1	1	1	42
31	-1	1	1	1	1	81
32	1	1	1	1	1	82

```
> summary(reactor1.lm)
Coefficients:
                Estimate Std. Error t value Pr(>|t|)
(Intercept)    6.550e+01        NaN     NaN      NaN
A             -6.875e-01        NaN     NaN      NaN
B              9.750e+00        NaN     NaN      NaN
C             -3.125e-01        NaN     NaN      NaN
D              5.375e+00        NaN     NaN      NaN
E             -3.125e+00        NaN     NaN      NaN
A:B            6.875e-01        NaN     NaN      NaN
A:C            3.750e-01        NaN     NaN      NaN
B:C            4.375e-01        NaN     NaN      NaN
A:D           -4.375e-01        NaN     NaN      NaN
B:D            6.625e+00        NaN     NaN      NaN
C:D            1.062e+00        NaN     NaN      NaN
A:E            6.250e-02        NaN     NaN      NaN
B:E            1.000e+00        NaN     NaN      NaN
C:E            4.375e-01        NaN     NaN      NaN
D:E           -5.500e+00        NaN     NaN      NaN
A:B:C          7.500e-01        NaN     NaN      NaN
A:B:D          6.875e-01        NaN     NaN      NaN
A:C:D         -3.750e-01        NaN     NaN      NaN
B:C:D          5.625e-01        NaN     NaN      NaN
A:B:E         -9.375e-01        NaN     NaN      NaN
A:C:E         -1.250e+00        NaN     NaN      NaN
B:C:E          6.250e-02        NaN     NaN      NaN
A:D:E          3.125e-01        NaN     NaN      NaN
B:D:E         -1.250e-01        NaN     NaN      NaN
C:D:E          6.250e-02        NaN     NaN      NaN
A:B:C:D       -2.946e-15        NaN     NaN      NaN
A:B:C:E        7.500e-01        NaN     NaN      NaN
A:B:D:E        3.125e-01        NaN     NaN      NaN
A:C:D:E        5.000e-01        NaN     NaN      NaN
B:C:D:E       -3.125e-01        NaN     NaN      NaN
```

Large effects are B, D, E, BD, and DE, all larger than 3. The largest 3- or 4-factor interaction is -1.25.

```
qqreactor1 <- function(){
  coef1 <- coef(reactor1.lm)[-1]
  qqnorm(coef1,main="Normal Q-Q Plot of 32 Run Results",pch="")
  qqline(coef1)
  tmp <- qqnorm(coef1,plot=F)
  text(tmp[[1]],tmp[[2]],names(coef1))
}
```

1. The first line gets the coefficients, minus the intercept.
2. The second line makes the qq normal plot, with blank plotting characters.
3. The third line draws the line through the points that look like a normal sample.
4. The fourth and fifth lines plot the names of the effects on top of where the point would have been plotted.

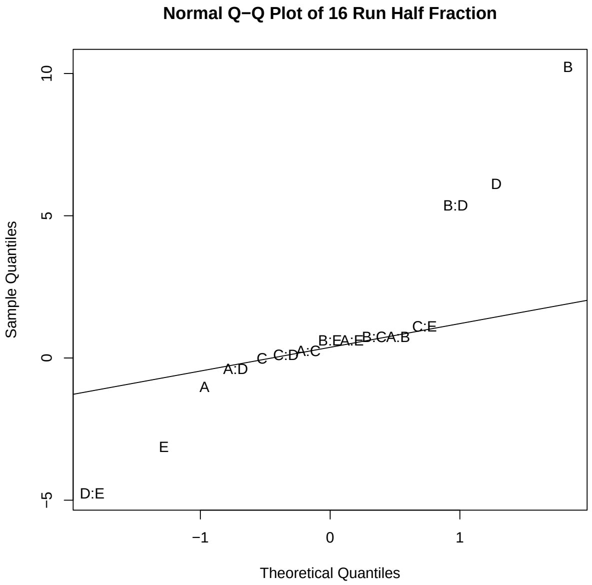


Plot of all 31 regression coefficients from 32-run design.

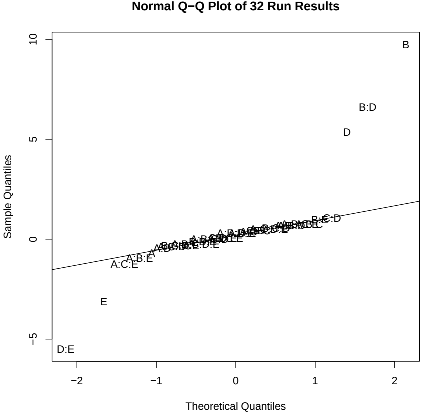
```
> reactor2 <- reactor1[with(reactor1,A*B*C*D==E),]
> reactor2
   run  A  B  C  D  E  y
2    2  1 -1 -1 -1 -1 53
3    3 -1  1 -1 -1 -1 63
5    5 -1 -1  1 -1 -1 53
8    8  1  1  1 -1 -1 61
9    9 -1 -1 -1  1 -1 69
12  12  1  1 -1  1 -1 93
14  14  1 -1  1  1 -1 60
15  15 -1  1  1  1 -1 95
17  17 -1 -1 -1 -1  1 56
20  20  1  1 -1 -1  1 65
22  22  1 -1  1 -1  1 55
23  23 -1  1  1 -1  1 67
26  26  1 -1 -1  1  1 45
27  27 -1  1 -1  1  1 78
29  29 -1 -1  1  1  1 49
32  32  1  1  1  1  1 82
```

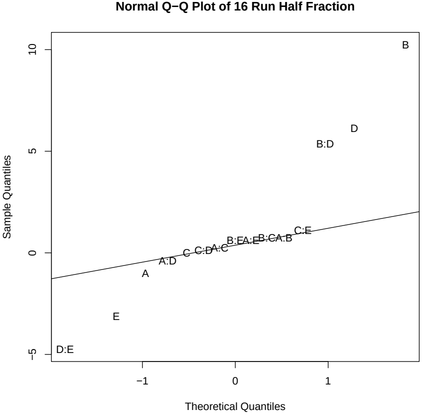
```
> summary(reactor2.lm)
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  6.525e+01        NaN     NaN      NaN
A           -1.000e+00        NaN     NaN      NaN
B            1.025e+01        NaN     NaN      NaN
C            2.437e-15        NaN     NaN      NaN
D            6.125e+00        NaN     NaN      NaN
E           -3.125e+00        NaN     NaN      NaN
A:B          7.500e-01        NaN     NaN      NaN
A:C          2.500e-01        NaN     NaN      NaN
B:C          7.500e-01        NaN     NaN      NaN
A:D         -3.750e-01        NaN     NaN      NaN
B:D          5.375e+00        NaN     NaN      NaN
C:D          1.250e-01        NaN     NaN      NaN
A:E          6.250e-01        NaN     NaN      NaN
B:E          6.250e-01        NaN     NaN      NaN
C:E          1.125e+00        NaN     NaN      NaN
D:E         -4.750e+00        NaN     NaN      NaN
```

This is the usual half-fraction in which **E** is identified with **ABCD**. The command at the top of the left column selects those runs, though they are not in the canonical order. Large effects are B, D, E, BD, and DE, all larger than 3. The largest remaining coefficient is 1.125.



Plot of all 15 regression coefficients from 16-run design. What did the other 16 runs buy them?


Normal Q−Q Plot of 32 Run Results
Sample Quantiles
Theoretical Quantiles
B
B:D
D
C:D
A:C:E
A:B:E
E
D:E


Normal Q−Q Plot of 16 Run Half Fraction
Sample Quantiles
Theoretical Quantiles
B
D
B:D
C:E
B:C
A:B
B:E
A:E
A:C
C:D
C
A:D
A
E
D:E

A few further points about this design can be noted.

- This half fraction is a 2_{V}^{5-1} design, so is of resolution 5 and therefore of projectivity 4, meaning that if any of the five factors is inert, the design is then a full factorial in the four remaining variables.
- This is so, since the defining relation is just $\mathbf{I} = \mathbf{ABCDE}$.
- This design is a factor screen of order $[16, 5, 4]$, meaning that it has 16 runs in 5 factors, and is projectivity 4.
- If all five factors were active, a second fraction of full foldover could be added, which would be two orthogonal blocks.
- It appears that factors **A** and **C** are completely inert in main effects and interactions, so we would get a replicated factorial in factors **B, D, E**

```
> summary(lm(y~B*D*E,data=reactor2))

Call:
lm(formula = y ~ B * D * E, data = reactor2)

Residuals:
   Min     1Q Median     3Q    Max 
  -4.5   -1.0    0.0    1.0    4.5 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  65.2500     0.7016  93.007 1.99e-13 ***
B            10.2500     0.7016  14.610 4.73e-07 ***
D             6.1250     0.7016   8.731 2.32e-05 ***
E            -3.1250     0.7016  -4.454 0.002127 ** 
B:D           5.3750     0.7016   7.661 5.95e-05 ***
B:E           0.6250     0.7016   0.891 0.398998    
D:E          -4.7500     0.7016  -6.771 0.000142 ***
B:D:E         0.2500     0.7016   0.356 0.730795    
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 2.806 on 8 degrees of freedom
Multiple R-squared:  0.9811,    Adjusted R-squared:  0.9645 
F-statistic: 59.28 on 7 and 8 DF,  p-value: 2.89e-06
```

Nodal 1/16 Fraction: 8 Factors in 16 Runs

This 2_{IV}^{8-4} design is of resolution 4, so it is of projectivity 3, meaning that it can form a replicated full factorial in any of the three factors from eight.This is a $[16, 8, 3]$ factor screen with 16 runs, 8 factors, and projectivity 3.

The example following has 8 (unnamed) factors $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{F}, \mathbf{G}, \mathbf{H}$ in paint manufacture for vehicles with $\mathbf{E} = \mathbf{ABC}$, $\mathbf{F} = \mathbf{ABD}$, $\mathbf{G} = \mathbf{ACD}$, and $\mathbf{H} = \mathbf{BCD}$. The responses were glossiness and abrasion resistance.

```
> paint
   run  A  B  C  D  E  F  G  H y1  y2
1    1 -1 -1 -1 -1 -1 -1 -1 -1 53 6.3
2    2  1 -1 -1 -1  1  1  1 -1 60 6.1
3    3 -1  1 -1 -1  1  1 -1  1 68 5.5
4    4  1  1 -1 -1 -1 -1  1  1 78 2.1
5    5 -1 -1  1 -1  1 -1  1  1 48 6.9
6    6  1 -1  1 -1 -1  1 -1  1 67 5.1
7    7 -1  1  1 -1 -1  1  1 -1 55 6.4
8    8  1  1  1 -1  1 -1 -1 -1 78 2.5
9    9 -1 -1 -1  1 -1  1  1  1 49 8.2
10  10  1 -1 -1  1  1 -1 -1  1 68 3.1
11  11 -1  1 -1  1  1 -1  1 -1 61 4.3
12  12  1  1 -1  1 -1  1 -1 -1 81 3.2
13  13 -1 -1  1  1  1  1 -1 -1 52 7.1
14  14  1 -1  1  1 -1 -1  1 -1 70 3.4
15  15 -1  1  1  1 -1 -1 -1  1 65 3.0
16  16  1  1  1  1  1  1  1  1 82 2.8
```

```
> coefb
              Gloss AbradeRes
(Intercept) 64.6875    4.7500
A            8.3125   -1.2125
B            6.3125   -1.0250
C           -0.0625   -0.1000
D            1.3125   -0.3625
E           -0.0625    0.0375
F           -0.4375    0.8000
G           -1.8125    0.2750
H            0.9375   -0.1625
A:B          0.4375    0.1375
A:C          1.3125    0.0125
A:D          0.9375   -0.0500
A:E         -0.9375    0.0500
A:F         -0.0625   -0.0375
A:G          1.3125   -0.2125
A:H         -0.1875   -0.1000
```

Factors **A** and **B** are important in glossiness and increase it. Factors **A**, **B**, and **F** are important in abrasion resistance, the first two decreasing it, and the third increasing. Factors **A** and **B** can be run at the high level, and the bad effect on abrasion resistance can be counteracted by increasing **F**, which helps abrasion resistance and does not damage glossiness significantly.

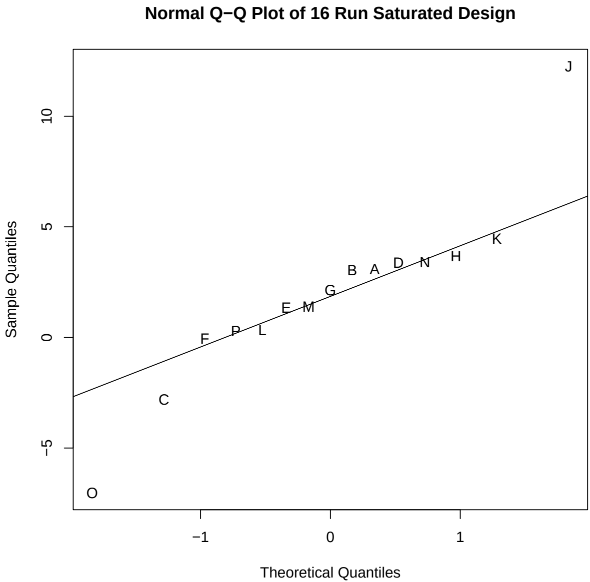
Nodal 1/64 Fraction: 15 Factors in 16 Runs

This 2^{15-11}_{III} design is of resolution 3, so it is of projectivity 2, meaning that it can form a replicated full factorial in any of the two factors from 15 (and a replicated 2^3 factorial from some sets of three factors).This is a $[16, 15, 2]$ factor screen with 16 runs, 15 factors, and projectivity 2.

The example following has 15 factors that might affect post-extrusion shrinkage of a speedometer cable casing.

```
> speedometer
   run  A  B  C  D  E  F  G  H  J  K  L  M  N  O  P  ave  var
1    1 -1 -1 -1 -1  1  1  1  1  1  1 -1 -1 -1 -1  1 48.5 16.3
2    2  1 -1 -1 -1 -1 -1 -1  1  1  1  1  1  1 -1 -1 57.5  4.3
3    3 -1  1 -1 -1 -1  1  1 -1 -1  1  1  1 -1  1 -1  8.8  2.9
4    4  1  1 -1 -1  1 -1 -1 -1 -1  1 -1 -1  1  1  1 17.5  3.0
5    5 -1 -1  1 -1  1 -1  1 -1  1 -1  1 -1  1  1 -1 19.5 20.3
6    6  1 -1  1 -1 -1  1 -1 -1  1 -1 -1  1 -1  1  1 14.5  5.7
7    7 -1  1  1 -1 -1 -1  1  1 -1 -1 -1  1  1 -1  1 22.5  7.0
8    8  1  1  1 -1  1  1 -1  1 -1 -1  1 -1 -1 -1 -1 17.5  9.0
9    9 -1 -1 -1  1  1  1 -1  1 -1 -1 -1  1  1  1 -1 12.5 19.7
10  10  1 -1 -1  1 -1 -1  1  1 -1 -1  1 -1 -1  1  1 12.0 58.0
11  11 -1  1 -1  1 -1  1 -1 -1  1 -1  1 -1  1 -1  1 45.5 13.7
12  12  1  1 -1  1  1 -1  1 -1  1 -1 -1  1 -1 -1 -1 53.5  0.3
13  13 -1 -1  1  1  1 -1 -1 -1 -1  1  1  1 -1 -1  1 17.0 10.7
14  14  1 -1  1  1 -1  1  1 -1 -1  1 -1 -1  1 -1 -1 27.5  3.7
15  15 -1  1  1  1 -1 -1 -1  1  1  1 -1 -1 -1  1 -1 34.2 22.9
16  16  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1 58.2 10.9
```

At each factor setting, 3000ft of extruded thermoplastic speedometer casing was produced (the smallest practicable) and four pieces of each casing were tested. The data above list the average of the four replicates and their variance (not log variance as stated in BHH). The original data are given in Quinlan (1985), a version of which is linked on the web site.



J, wire braid type and **O**, wire diameter were the most important, followed by **C**, liner die.

```
> summary(lm(ave~C+J+O,data=speedometer))

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   29.169      2.651  11.003 1.26e-07 ***
C             -2.806      2.651  -1.059 0.310646
J             12.256      2.651   4.623 0.000587 ***
O             -7.019      2.651  -2.648 0.021274 *
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 10.6 on 12 degrees of freedom
Multiple R-squared:  0.7109,    Adjusted R-squared:  0.6386
F-statistic: 9.835 on 3 and 12 DF,  p-value: 0.001485
```

The residual mean square is 112.45, which is an estimate of the variance between conceptual replicates at the same conditions. The mean of the within-sample variances is 13.025, so there is a lot of variation between 3000-foot pieces of speedometer casing that is not represented by the variation within the piece. This is very common, and shows the great importance of choosing the right comparison variance. The multiple pieces tested on each length of cable did not increase the accuracy of the result by much!

A true set of four replicates would have to have been done on four separate 3000-foot pieces of casing embedded in a fully randomized order of $16 \times 4 = 64$ runs. What would happen if we used the four individual data values for each factor combination? The residual variance in R would be $s^2 = 13.025$ exactly and the estimated variance of one of the effects would be $s^2/8 + s^2/8 = s^2/4 = 13.025/4 = 3.256$, so the standard error would be 1.80 with 48 df and the standard error of a regression coefficient would 0.90 (compare to 2.65 from the regression on the last slide). This would mean that coefficients larger than about 1.8 would be significant, and that would include A, B, C, D, G, H, J, K, N, O. Only 2 or 3 of these 10 are likely real, and the rest are an artifact of using variation within a casing to compare to variation between casings, which is clearly larger.