

Survival Data and Methods

David M. Rocke

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Mean Residual Life

The mean lifetime with a survival distribution $f(x)$ is

$$\int_0^{\infty} xf(x)dx$$

For the exponential distribution we know that the mean is λ^{-1} . The mean residual life after survival to time x is

$$mrl(x) = S^{-1}(x) \int_x^{\infty} (u - x)f(u)du$$

For the exponential, the mean residual life is also λ^{-1}

Period Life Table, 2019

Exact age	Male			Female		
	Death probability ^a	Number of lives ^b	Life expectancy	Death probability ^a	Number of lives ^b	Life expectancy
0	0.006081	100,000	76.23	0.005046	100,000	81.28
1	0.000425	99,392	75.69	0.000349	99,495	80.69
2	0.000260	99,350	74.73	0.000212	99,461	79.72
3	0.000194	99,324	73.75	0.000166	99,440	78.74
4	0.000154	99,305	72.76	0.000137	99,423	77.75
5	0.000142	99,289	71.77	0.000122	99,409	76.76
6	0.000135	99,275	70.78	0.000111	99,397	75.77
7	0.000127	99,262	69.79	0.000103	99,386	74.78
8	0.000117	99,249	68.80	0.000098	99,376	73.79
9	0.000104	99,238	67.81	0.000095	99,366	72.79
10	0.000097	99,227	66.81	0.000096	99,357	71.80
11	0.000106	99,218	65.82	0.000102	99,347	70.81
12	0.000145	99,207	64.83	0.000116	99,337	69.81
13	0.000220	99,193	63.84	0.000139	99,326	68.82
14	0.000324	99,171	62.85	0.000170	99,312	67.83
15	0.000437	99,139	61.87	0.000204	99,295	66.84
16	0.000552	99,096	60.90	0.000240	99,275	65.86
17	0.000676	99,041	59.93	0.000278	99,251	64.87
18	0.000806	98,974	58.97	0.000319	99,223	63.89
19	0.000939	98,894	58.02	0.000360	99,192	62.91

- The 2019 US standard mortality table estimates the expectation of life of females at birth as 81.28 years.
- At age 50, 95.6% of US females are still alive.
- The mean residual life at age 50 is 33.51 years (age $50 + 33.51 = 83.51$). At age 83, 56.4% are still alive.
- In 1850 an estimate of the expectation of life at birth for females is 39.4 years. At age 1, it is $1 + 49.3 = 50.3$
- But 44.7% of females lived to age 50 and the further expectation of life was 20.4 years, so to age 70.4. About 24% lived to age 70 and 10% to age 80.
- So it was not rare to live beyond age 39.

Actuarial Life Tables

- The text describes cohort life tables, which are constructed by following a cohort from initiation of the study until all events have occurred. We will not pursue this further—we usually use the raw data for analysis.
- Another form of life table is an actuarial life table which is constructed from death rates in a particular year and present life data for a hypothetical population in which at every age the death rates were the same as in the year the table was constructed.

Actuarial Life Tables

- This applies as if a (say) female person born 1/1/2019 would have a chance of dying between age 50 and 51 equal to the chance a female person born 1/1/1969 had of dying between the age of 50 and 51 even though the hypothetical event would take place between 1/1/2069 and 12/31/2069.
- In reality, a 50 year old person in 2019 would have a death rate depending on the current year (time effect) and the year of birth (cohort effect).
- The cohort effect in 2069 for those born in 2019 and the time effect for those aged 50 in 2069 are both unknown.

Actuarial Life Tables

- This is however the best we can do, and is in any case standard.
- Announcements that the life expectancy in the US dropped in 2020 by one year is, however, not useful. This comes from applying the excess mortality due to COVID to each future year of someone born in 2020, which seems unlikely to be correct.
- These tables have substantial practical importance and it is unclear what SSA and others will make of this.

How much is a pension worth?

- A pension or another annuity is a promise to pay an amount to someone (say) each month as long as they are alive.
- There are complications, and the tables are probably given in ages in years and months, but we can get the general idea using annual table.
- The value depends on the periodic amount, the age at initiation, the vector of annual survival probabilities, and the discount rate.

How much is a pension worth?

$$\begin{aligned} V &= \sum_{t=1}^{\infty} M(1+i)^{-t} \prod_{s=0}^{t-1} P(A+s) \\ &= M(1+i)^{-1}P(A) + M(1+i)^{-2}P(A)P(A+1) + \dots \end{aligned}$$

where V is the value of the annuity, M is the annual amount paid at the end of each year, i is the discount rate, A is the age at initiation, $P(x)$ is the chance of surviving one year at age x

60	0.011447	85,506	21.77	0.006880	91,396	24.80
61	0.012352	84,527	21.02	0.007454	90,767	23.96
62	0.013248	83,483	20.28	0.008006	90,091	23.14

The value of the first two end of year payments of \$1000 to a male person age 60 conditional on surviving until the end of each year with a discount rate of 5% is

$$\begin{aligned} \$1000(1 - 0.011447)/(1.05) &= \\ \$1000(0.988553)(0.95238) &= \$941.48 \end{aligned}$$

$$\begin{aligned} \$1000(1 - 0.011447)(1 - 0.012352)/(1.05)^2 &= \\ \$1000(0.988553)(0.987648)/(1.05)^2 &= \\ \$1000(0.97634)(0.90703) &= \$885.57 \end{aligned}$$

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The value of the first end of year payment of \$1000 to a male person age 60 and a female person aged 61 conditional on at least one of them surviving until the end of the year, assuming independence, with a discount rate of 5% is

$$\begin{aligned} \$1000(1 - 0.011447 \times 0.007454)/(1.05) &= \\ \$1000(0.999991)/(0.95238) &= \$952.30 \end{aligned}$$

The event in which the second year is paid is already complicated consisting of 1) both survive 2 years; 2) the man survives 0 or 1 year and the woman survives 2 years; 3) the woman survives 0 or 1 year and the man survives 2 years. There are tables and computer programs to help with this, but you will do one such calculation on the homework.

Other Parametric Survival Distributions

- Any density on $[0, \infty)$ can be a survival distribution, but the most useful ones are all skew right.
- The commonest generalization of the exponential is the Weibull.
- Other common choices are the gamma, log-normal, log-logistic, Gompertz, inverse Gaussian, and Pareto.
- Most of what we do going forward is non-parametric or semi-parametric, but sometimes these parametric distributions provide a useful approach and we will look at them later in the quarter.

Weibull Distribution

$$f(x) = \alpha \lambda x^{\alpha-1} e^{-\lambda x^\alpha}$$

$$h(x) = \alpha \lambda x^{\alpha-1}$$

$$S(x) = e^{-\lambda x^\alpha}$$

$$E(X) = \Gamma(1 + 1/\alpha) / \lambda^{1/\alpha}$$

When $\alpha = 1$ this is the exponential. When $\alpha > 1$ the hazard is increasing and when $\alpha < 1$ the hazard is decreasing. This provides more flexibility than the exponential.

Nonparametric Survival Analysis

- Mostly, we work without a parametric model.
- The first task is to estimate a survival function from data listing survival times, and censoring times for right-censored data.
- For example one patient may have relapsed at 10 months. Another might have been followed for 32 months without a relapse having occurred (censored).
- The minimum information we need for each patient is a time and a censoring variable which is 1 if the event occurred at the indicated time and 0 if this is a censoring time.

KM drug6mp Data

This is from a clinical trial in 1963 for 6-MP treatment vs. placebo for Acute Leukemia in 42 children. Pairs of children matched by remission status at the time of treatment (1 = partial or 2 = complete) and randomized to 6-MP or placebo. Followed until relapse or end of study. All of the placebo group relapsed, but some of the 6-MP group were censored (which means they were still in remission).

6-MP = 6-Mercaptopurine (Purinethol) is an anti-cancer (“antineoplastic” or “cytotoxic”) chemotherapy drug used currently for Acute Lymphoblastic Leukemia (ALL). It is classified as an antimetabolite.

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```
> library(KMsurv)
> data(drug6mp)
> drug6mp
```

	pair	remstat	t1	t2	relapse
1	1	1	1	10	1
2	2	2	22	7	1
3	3	2	3	32	0

KM drug6mp Data

drug6mp data

Description

The drug6mp data frame has 21 rows and 5 columns.

Format

This data frame contains the following columns:

pair	pair number
remstat	Remission status at randomization (1=partial, 2=complete)
t1	Time to relapse for placebo patients, months
t2	Time to relapse for 6-MP patients, months
relapse	Relapse indicator (0=censored, 1=relapse) for 6-MP patients

Descriptive Statistics

- The average time in each group is not useful. Some of the 6-MP patients have not relapsed at the time recorded, while all of the placebo patients have relapsed.
- The median time is not really useful either because so many of the 6-MP patients have not relapsed (12/21).
- Both are biased down in the 6-MP group. Remember that lower times are worse since they indicate sooner recurrence.

Descriptive Statistics

- We can compute the average hazard rate, which is the estimate of the exponential parameter: number of relapses divided by the sum of the times.
- For the placebo, that is just the reciprocal of the mean time = $1/8.667 = 0.115$.
- For the 6-MP group this is $9/359 = 0.025$
- The estimated average hazard in the placebo group is 4.6 times as large (if the hazard is constant over time).

The Kaplan-Meier Product Limit Estimator

- The estimated survival function for the placebo patients is easy to compute. For any time t in months, $S(t)$ is the fraction of patients with times greater than t .
- For the 6-MP patients, we cannot ignore the censored data because we know that the time to relapse is greater than the censoring time.

The Kaplan-Meier Product Limit Estimator

- For any time t in months, we know that 6-MP patients with times greater than t have not relapsed, and those with relapse time less than t have relapsed, but we don't know if patients with censored time less than t have relapsed or not.
- The procedure we usually use is the Kaplan-Meier product-limit estimator of the survival function. We will also learn the Nelson-Aalen procedure which is preferred for some uses.

- The Kaplan-Meier estimator is a step function (like the empirical cdf), which changes value only at the event times, not at the censoring times.
- At each event time t , we compute the at-risk group size Y , which is all those observations whose event time or censoring time is at least t .
- If d of the observations have an event time (not a censoring time) of t , then the group of survivors immediately following time t is reduced by the fraction

$$\frac{Y - d}{Y} = 1 - \frac{d}{Y}$$

If the event times are t_i with events per time of d_i ($1 \leq i \leq k$), then

$$\hat{S}(t) = \prod_{t_i < t} [1 - d_i / Y_i]$$

where Y_i is the set of observations whose time (event or censored) is $\geq t_i$, the group at risk at time t_i .

If there are no censored data, and there are n data points, then just after (say) the third event time

$$\begin{aligned}\hat{S}(t) &= \prod_{t_i < t} [1 - d_i / Y_i] \\ &= \left[\frac{n - d_1}{n} \right] \left[\frac{n - d_1 - d_2}{n - d_1} \right] \left[\frac{n - d_1 - d_2 - d_3}{n - d_1 - d_2} \right] \\ &= \frac{n - d_1 - d_2 - d_3}{n}\end{aligned}$$

the usual empirical cdf estimate.


```
require(KMsurv)
data(drug6mp)
plot(survfit(Surv(drug6mp$t2,drug6mp$relapse)~1))
title("Kaplan-Meier Survival Curve for 6-MP Patients")

time12 <- c(drug6mp$t1,drug6mp$t2)
cens12 <- c(rep(1,21),drug6mp$relapse)
treat12 <- rep(1:2,each=21)
pairs12 <- rep(1:21,2)

plot(survfit(Surv(time12,cens12)~treat12),col=1:2)
title("Kaplan-Meier Survival Curve for 6-MP and Placebo Patients")

plot(survfit(Surv(time12,cens12)~treat12),conf.int=T,col=1:2)
title("Kaplan-Meier Survival Curve for 6-MP and Placebo Patients")
```

Time	At Risk	Relapses	Censored	KM Factor	KM Curve
6	21	3	1	0.857	0.857
7	17	1	0	0.941	0.807
9	16	0	1	1	0.807
10	15	1	1	0.933	0.753
11	13	0	1	1	0.753
13	12	1	0	0.917	0.690
16	11	1	0	0.909	0.627
17	10	0	1	1	0.627
19	9	0	1	1	0.627
20	8	0	1	1	0.627
22	7	1	0	0.857	0.538
23	6	1	0	0.833	0.448
25	5	0	1	1	0.448
32	4	0	2	1	0.448
34	2	0	1	1	0.448
35	1	0	1	1	0.448

For the 6-MP patients at time 6 months, there are 21 patients at risk. At $t = 6$ there are 3 relapses and 1 censored observations. The Kaplan-Meier factor is $(21 - 3)/21 = 0.857$. The number at risk for the next time ($t = 7$) is $21 - 3 - 1 = 17$.

At time 7 months, there are 17 patients at risk. At $t = 7$ there is 1 relapse and 0 censored observations. The Kaplan-Meier factor is $(17 - 1)/17 = 0.941$. The Kaplan Meier estimate is $0.857 \times 0.941 = 0.807$. The number at risk for the next time ($t = 9$) is $17 - 1 = 16$.

```
time12 <- c(drug6mp$t1,drug6mp$t2)
cens12 <- c(rep(1,21),drug6mp$relapse)
treat12 <- rep(1:2,each=21)
pairs12 <- rep(1:21,2)
```

```
print(survdiff(Surv(time12,cens12)~treat12))
```

	N	Observed	Expected	$(O-E)^2/E$	$(O-E)^2/V$
treat12=1	21	21	10.7	9.77	16.8
treat12=2	21	9	19.3	5.46	16.8

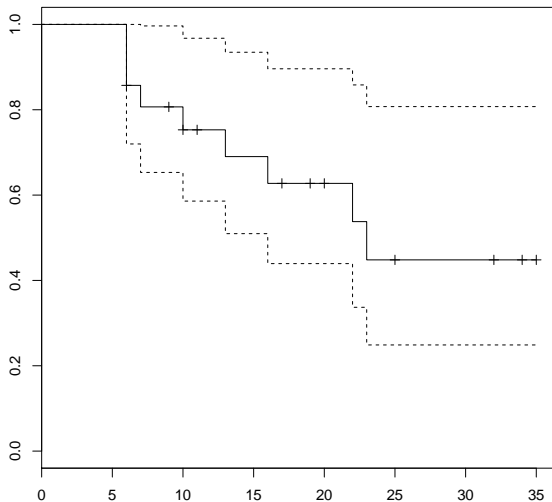
Chisq= 16.8 on 1 degrees of freedom, p= 4.17e-05

```
print(survdiff(Surv(time12,cens12)~treat12+strata(pairs12)))
```

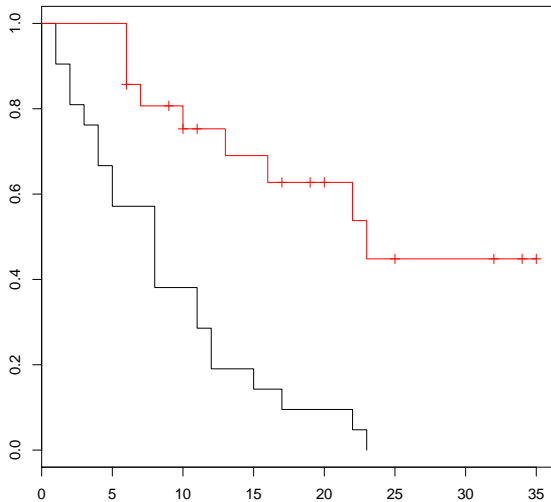
	N	Observed	Expected	$(O-E)^2/E$	$(O-E)^2/V$
treat12=1	21	21	13.5	4.17	10.7
treat12=2	21	9	16.5	3.41	10.7

Chisq= 10.7 on 1 degrees of freedom, p= 0.00106

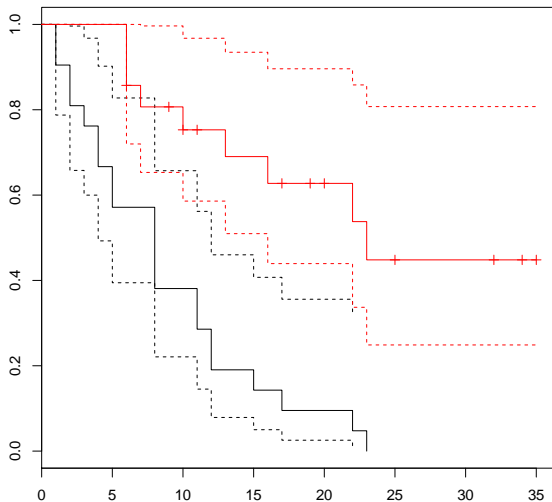
Kaplan–Meier Survival Curve for 6–MP Patients



Kaplan–Meier Survival Curve for 6-MP and Placebo Patients



Kaplan–Meier Survival Curve for 6-MP and Placebo Patients



Package Survival

Surv

Create a survival object, usually used as a response variable in a model formula.

Usage

```
Surv(time, event)
```

Arguments

time for right censored data, this is the follow up time.

event The status indicator, normally 0=alive, 1=dead.
Also TRUE/FALSE (TRUE = death) or 1/2 (2=death).
The event indicator can be omitted,
in which case all subjects are assumed to have an event.

```
Surv(drug6mp$t2, drug6mp$relapse)
```


Package Survival

survfit

This function creates survival curves from either a formula (e.g. the Kaplan-Meier), a previously fitted Cox model, or a previously fitted accelerated failure time model.

Usage

```
survfit(formula, ...)
```

Arguments

formula either a formula or a previously fitted model

```
plot(survfit(Surv(drug6mp$t2,drug6mp$relapse)~1))  
plot(survfit(Surv(time12,cens12)~treat12))
```

Package Survival

`survdiff`

Tests if there is a difference between two or more survival curves.

Usage

```
survdiff(formula, data, subset, na.action, rho=0)
```

Arguments

`formula` a formula expression as for other survival models, of the form `Surv(time, status) ~ predictors`. A strata term may be used to produce a stratified test.

`rho` Type of test. Default is the Mantel-Haenszel test.

```
-----  
print(survdiff(Surv(time12,cens12)~treat12))  
print(survdiff(Surv(time12,cens12)~treat12+strata(pairs12)))
```