CHAPTER 13

13.1 (a) The function can be differentiated to give

\[ f'(x) = -2x + 8 \]

This function can be set equal to zero and solved for \( x = 8/2 = 4 \). The derivative can be differentiated to give the second derivative

\[ f''(x) = -2 \]

Because this is negative, it indicates that the function has a maximum at \( x = 4 \).

(b) Using Eq. 13.7

\[
\begin{align*}
  x_0 &= 0 & f(x_0) &= -12 \\
  x_1 &= 2 & f(x_1) &= 0 \\
  x_2 &= 6 & f(x_2) &= 0 \\
  x_3 &= \frac{-12(4 - 36) + 0(36 - 0) + 0(0 - 4)}{2(-12)(2 - 6) + 2(0)(6 - 0) + 2(0)(0 - 2)} = 4
\end{align*}
\]

13.2 (a) The function can be plotted

(b) The function can be differentiated twice to give

\[ f''(x) = -45x^4 - 24x^2 \]

Thus, the second derivative will always be negative and hence the function is concave for all values of \( x \).

(c) Differentiating the function and setting the result equal to zero results in the following roots problem to locate the maximum

\[ f'(x) = 0 = -9x^5 - 8x^3 + 12 \]

A plot of this function can be developed
A technique such as bisection can be employed to determine the root. Here are the first few iterations:

<table>
<thead>
<tr>
<th>iteration</th>
<th>$x_l$</th>
<th>$x_u$</th>
<th>$x_r$</th>
<th>$f(x_l)$</th>
<th>$f(x_r)$</th>
<th>$f(x_l) \times f(x_r)$</th>
<th>$\varepsilon_u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.00000</td>
<td>2.00000</td>
<td>1.00000</td>
<td>12</td>
<td>-5</td>
<td>-60.0000</td>
<td>100.00%</td>
</tr>
<tr>
<td>2</td>
<td>0.00000</td>
<td>1.00000</td>
<td>0.50000</td>
<td>12</td>
<td>10.71875</td>
<td>128.6250</td>
<td>100.00%</td>
</tr>
<tr>
<td>3</td>
<td>0.50000</td>
<td>1.00000</td>
<td>0.75000</td>
<td>10.71875</td>
<td>6.489258</td>
<td>69.5567</td>
<td>33.33%</td>
</tr>
<tr>
<td>4</td>
<td>0.75000</td>
<td>1.00000</td>
<td>0.87500</td>
<td>6.489258</td>
<td>2.024445</td>
<td>13.1371</td>
<td>14.29%</td>
</tr>
<tr>
<td>5</td>
<td>0.87500</td>
<td>1.00000</td>
<td>0.93750</td>
<td>2.024445</td>
<td>-1.10956</td>
<td>-2.2463</td>
<td>6.67%</td>
</tr>
</tbody>
</table>

The approach can be continued to yield a result of $x = 0.91692$.

13.3 First, the golden ratio can be used to create the interior points,

\[
d = \frac{\sqrt{5} - 1}{2} (2 - 0) = 1.2361
\]

\[x_1 = 0 + 1.2361 = 1.2361\]

\[x_2 = 2 - 1.2361 = 0.7639\]

The function can be evaluated at the interior points

\[f(x_2) = f(0.7639) = 8.1879\]

\[f(x_1) = f(1.2361) = 4.8142\]

Because $f(x_2) > f(x_1)$, the maximum is in the interval defined by $x_1$, $x_2$, and $x_1$, where $x_2$ is the optimum. The error at this point can be computed as

\[\varepsilon_u = (1 - 0.61803) \left| \frac{2 - 0}{0.7639} \right| \times 100\% = 100\%\]

For the second iteration, $x_1 = 0$ and $x_u = 1.2361$. The former $x_2$ value becomes the new $x_1$, that is, $x_1 = 0.7639$ and $f(x_1) = 8.1879$. The new values of $d$ and $x_2$ can be computed as

\[d = \frac{\sqrt{5} - 1}{2} (1.2361 - 0) = 0.7639\]

\[x_2 = 1.2361 - 0.7639 = 0.4721\]

The function evaluation at $f(x_2) = 5.5496$. Since this value is less than the function value at $x_1$, the maximum is in the interval prescribed by $x_2$, $x_1$, and $x_u$. The process can be repeated and all three iterations summarized as
13.4 First, the function values at the initial values can be evaluated

\[ f(x_0) = f(0) = 0 \]
\[ f(x_1) = f(1) = 8.5 \]
\[ f(x_2) = f(2) = -104 \]

and substituted into Eq. (13.7) to give,

\[ x_3 = \frac{0(1^2 - 2^2) + 8.5(2^2 - 0^2) + (-104)(0^2 - 1^2)}{2(0)(1 - 2) + 2(8.5)(2 - 0) + 2(-104)(0 - 1)} = 0.570248 \]

which has a function value of \( f(0.570248) = 6.5799 \). Because the function value for the new point is lower than for the intermediate point \( (x_1) \) and the new \( x \) value is to the left of the intermediate point, the lower guess \( (x_0) \) is discarded. Therefore, for the next iteration,

\[ f(x_0) = f(0.570248) = 6.6799 \]
\[ f(x_1) = f(1) = 8.5 \]
\[ f(x_2) = f(2) = -104 \]

which can be substituted into Eq. (13.7) to give \( x_3 = 0.812431 \), which has a function value of \( f(0.812431) = 8.446523 \). At this point, an approximate error can be computed as

\[ \varepsilon_a = \left| \frac{0.81243 - 0.570248}{0.81243} \right| \times 100\% = 29.81\% \]

The process can be repeated, with the results tabulated below:

<table>
<thead>
<tr>
<th>( i )</th>
<th>( x_0 )</th>
<th>( f(x_0) )</th>
<th>( x_1 )</th>
<th>( f(x_1) )</th>
<th>( x_2 )</th>
<th>( f(x_2) )</th>
<th>( x_3 )</th>
<th>( f(x_3) )</th>
<th>( \varepsilon_a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.00000</td>
<td>0.00000</td>
<td>1.00000</td>
<td>8.50000</td>
<td>2.00000</td>
<td>-104</td>
<td>0.57025</td>
<td>6.57991</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.57025</td>
<td>6.57991</td>
<td>1.00000</td>
<td>8.50000</td>
<td>2.00000</td>
<td>-104</td>
<td>0.81243</td>
<td>8.44652</td>
<td>29.81%</td>
</tr>
<tr>
<td>3</td>
<td>0.81243</td>
<td>8.44652</td>
<td>1.00000</td>
<td>8.50000</td>
<td>2.00000</td>
<td>-104</td>
<td>0.90772</td>
<td>8.69575</td>
<td>10.50%</td>
</tr>
</tbody>
</table>

Thus, after 3 iterations, the result is converging on the true value of \( f(x) = 8.69793 \) at \( x = 0.91692 \).

13.5 The first and second derivatives of the function can be evaluated as

\[ f'(x) = -9x^5 - 8x^3 + 12 \]
\[ f''(x) = -45x^4 - 24x^2 \]

which can be substituted into Eq. (13.8) to give

\[ x_{i+1} = x_i - \frac{-9x_i^5 - 8x_i^3 + 12}{-45x_i^4 - 24x_i^2} \]
Substituting the initial guess yields
\[ x_{r+1} = 2 - \frac{-9(2^3) - 8(2^3) + 12}{-45(2^4) - 24(2^2)} = 2 - \frac{-340}{-816} = 1.583333 \]
which has a function value of \(-17.2029\). The second iteration gives
\[ x_{r+1} = 1.583333 - \frac{-9(1.583333^3) - 8(1.583333^3) + 12}{-45(1.583333^4) - 24(1.583333^2)} = 1.583333 - \frac{-109.313}{-342.981} = 1.26462 \]
which has a function value of \(3.924617\). At this point, an approximate error can be computed as
\[ \varepsilon = \left| \frac{1.26462 - 1.583333}{1.26462} \right| \times 100\% = 26.316\% \]

The process can be repeated, with the results tabulated below:

<table>
<thead>
<tr>
<th>(i)</th>
<th>(x)</th>
<th>(f(x))</th>
<th>(f'(x))</th>
<th>(f''(x))</th>
<th>(\varepsilon)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>-104</td>
<td>-340</td>
<td>-816</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.583333</td>
<td>-17.2029</td>
<td>-109.313</td>
<td>-342.981</td>
<td>26.316%</td>
</tr>
<tr>
<td>2</td>
<td>1.26462</td>
<td>3.924617</td>
<td>-33.2898</td>
<td>-153.476</td>
<td>25.202%</td>
</tr>
<tr>
<td>3</td>
<td>1.047716</td>
<td>8.178616</td>
<td>-8.56281</td>
<td>-80.5683</td>
<td>20.703%</td>
</tr>
</tbody>
</table>

Thus, within five iterations, the result is converging on the true value of \(f(x) = 8.69793\) at \(x = 0.91692\).

13.6 (a) First, the golden ratio can be used to create the interior points,
\[ d = \frac{\sqrt{5} - 1}{2}(4 - (-2)) = 3.7082 \]
\[ x_1 = -2 + 3.7082 = 1.7082 \]
\[ x_2 = 4 - 3.7082 = 0.2918 \]
The function can be evaluated at the interior points
\[ f(x_2) = f(0.2918) = 1.04156 \]
\[ f(x_1) = f(1.7082) = 5.00750 \]
Because \(f(x_1) > f(x_2)\), the maximum is in the interval defined by \(x_2, x_1\) and \(x_u\) where \(x_1\) is the optimum. The error at this point can be computed as
\[ \varepsilon = (1 - 0.61803) \left| \frac{4 - (-2)}{1.7082} \right| \times 100\% = 134.16\% \]

The process can be repeated and all the iterations summarized as

<table>
<thead>
<tr>
<th>(i)</th>
<th>(x_1)</th>
<th>(f(x_1))</th>
<th>(x_2)</th>
<th>(f(x_2))</th>
<th>(x_1)</th>
<th>(f(x_1))</th>
<th>(x_u)</th>
<th>(f(x_u))</th>
<th>(d)</th>
<th>(x_{opt})</th>
<th>(\varepsilon)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-2.0000</td>
<td>-29.6000</td>
<td>0.2918</td>
<td>1.0416</td>
<td>1.7082</td>
<td>5.0075</td>
<td>4.0000</td>
<td>-12.8000</td>
<td>3.7082</td>
<td>1.7082</td>
<td>134.16%</td>
</tr>
<tr>
<td>2</td>
<td>0.2918</td>
<td>1.0416</td>
<td>1.7082</td>
<td>5.0075</td>
<td>2.5836</td>
<td>5.6474</td>
<td>4.0000</td>
<td>-12.8000</td>
<td>2.2918</td>
<td>2.5836</td>
<td>54.82%</td>
</tr>
<tr>
<td>3</td>
<td>1.7082</td>
<td>5.0075</td>
<td>2.5836</td>
<td>5.6474</td>
<td>3.1246</td>
<td>2.9361</td>
<td>4.0000</td>
<td>-12.8000</td>
<td>1.4164</td>
<td>2.5836</td>
<td>33.88%</td>
</tr>
</tbody>
</table>

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(b) First, the function values at the initial values can be evaluated

\[ f(x_0) = f(1.75) = 5.1051 \]
\[ f(x_1) = f(2) = 5.6 \]
\[ f(x_2) = f(2.5) = 5.7813 \]

and substituted into Eq. (13.7) to give,

\[ x_3 = \frac{5.1051(2^2 - 2.5^2) + 5.6(2.5^2 - 1.75^2) + 5.7813(1.75^2 - 2^2)}{2(5.1051)(2 - 2.5) + 2(5.6)(2.5 - 1.75) + 2(5.7813)(1.75 - 2)} = 2.3341 \]

Second iteration:

\[ f(x_0) = f(2) = 5.6 \]
\[ f(x_1) = f(2.5) = 5.7813 \]
\[ f(x_2) = f(2.3341) = 5.8852 \]

which can be substituted into Eq. (13.7) to give \( x_3 = 2.3112 \), which has a function value of \( f(2.3112) = 5.8846 \). At this point, an approximate error can be computed as

\[ \epsilon_a = \left| \frac{2.3112 - 2.3341}{2.3112} \right| \times 100\% = 0.99\% \]

The process can be repeated, with the results tabulated below:

<table>
<thead>
<tr>
<th>(i)</th>
<th>(x_0)</th>
<th>(f(x_0))</th>
<th>(x_1)</th>
<th>(f(x_1))</th>
<th>(x_2)</th>
<th>(f(x_2))</th>
<th>(x_3)</th>
<th>(f(x_3))</th>
<th>(\epsilon_a)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.7500</td>
<td>5.1051</td>
<td>2.0000</td>
<td>5.6000</td>
<td>2.5000</td>
<td>5.7813</td>
<td>2.3341</td>
<td>5.8852</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2.0000</td>
<td>5.6000</td>
<td>2.5000</td>
<td>5.7813</td>
<td>2.3341</td>
<td>5.8852</td>
<td>2.3112</td>
<td>5.8846</td>
<td>0.99%</td>
</tr>
<tr>
<td>3</td>
<td>2.5000</td>
<td>5.7813</td>
<td>2.3341</td>
<td>5.8852</td>
<td>2.3112</td>
<td>5.8846</td>
<td>2.3260</td>
<td>5.8853</td>
<td>0.64%</td>
</tr>
<tr>
<td>4</td>
<td>2.3341</td>
<td>5.8852</td>
<td>2.3112</td>
<td>5.8846</td>
<td>2.3260</td>
<td>5.8853</td>
<td>2.3263</td>
<td>5.8853</td>
<td>0.01%</td>
</tr>
</tbody>
</table>

Thus, after 4 iterations, the result is converging rapidly on the true value of \( f(x) = 5.8853 \) at \( x = 2.3263 \).

(c) The first and second derivatives of the function can be evaluated as

\[ f'(x) = 4 - 3.6x + 3.6x^2 - 1.2x^3 \]
\[ f''(x) = -3.6 + 7.2x - 3.6x^2 \]

which can be substituted into Eq. (13.8) to give
\[ x_{i+1} = x_i - \frac{4 - 3.6x_i + 3.6x_i^2 - 1.2x_i^3}{-3.6 + 7.2x_i - 3.6x_i^2} = 3 - \frac{-6.8}{-14.4} = 2.5278 \]

which has a function value of 5.7434. The second iteration gives 2.3517, which has a function value of 5.8833. At this point, an approximate error can be computed as \( \varepsilon_a = 18.681\% \). The process can be repeated, with the results tabulated below:

<table>
<thead>
<tr>
<th>( i )</th>
<th>( x )</th>
<th>( f(x) )</th>
<th>( f'(x) )</th>
<th>( f''(x) )</th>
<th>( \varepsilon_a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.0000</td>
<td>3.9000</td>
<td>-6.8000</td>
<td>-14.4000</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2.5278</td>
<td>5.7434</td>
<td>-1.4792</td>
<td>-8.4028</td>
<td>18.681%</td>
</tr>
<tr>
<td>2</td>
<td>2.3517</td>
<td>5.8833</td>
<td>-0.1639</td>
<td>-6.5779</td>
<td>7.485%</td>
</tr>
<tr>
<td>3</td>
<td>2.3268</td>
<td>5.8853</td>
<td>-0.0030</td>
<td>-6.3377</td>
<td>1.071%</td>
</tr>
<tr>
<td>4</td>
<td>2.3264</td>
<td>5.8853</td>
<td>0.0000</td>
<td>-6.3332</td>
<td>0.020%</td>
</tr>
</tbody>
</table>

Thus, within four iterations, the result is converging on the true value of \( f(x) = 5.8853 \) at \( x = 2.3264 \).

13.7 The function can be differentiated twice to give

\[
\begin{align*}
  f'(x) &= -4x^3 - 6x^2 - 16x - 5 \\
  f''(x) &= -12x^2 - 12x - 16
\end{align*}
\]

which is negative for \(-2 \leq x \leq 1\). This suggests that an optimum in the interval would be a maximum. A graph of the original function shows a maximum at about \( x = -0.35 \).

13.8 (a) First, the golden ratio can be used to create the interior points,

\[
d = \frac{\sqrt{5} - 1}{2} - (1 - (-2)) = 1.8541
\]

\[
x_1 = -2 + 1.8541 = -0.1459
\]

\[
x_2 = 1 - 1.8541 = -0.8541
\]

The function can be evaluated at the interior points

\[
\begin{align*}
  f(x_2) &= f(-0.8541) = -0.8514 \\
  f(x_1) &= f(-0.1459) = 0.5650
\end{align*}
\]

Because \( f(x_1) < f(x_2) \), the maximum is in the interval defined by \( x_2, x_1 \) and \( x_o \) where \( x_1 \) is the optimum. The error at this point can be computed as
14.7 The partial derivatives can be evaluated at the initial guesses, \( x = 0 \) and \( y = 0 \),

\[
\frac{\partial f}{\partial x} = 3.5 + 2x - 4x^3 - 2y = 3.5 + 2(0) - 4(0)^3 - 2(0) = 3.5 \\
\frac{\partial f}{\partial y} = 2 - 2x - 2y = 2 - 2(0) - 2(0) = 2 \\
f(0 + 3.5h, 0 + 2h) = 16.25h - 5.75h^2 - 150.06h^4 \\
g'(h) = 16.25 - 11.5h - 600.25h^3
\]

The root of this equation can be determined by bisection. Using initial guesses of \( h = 0 \) and 1 yields a root of \( h^* = 0.27893 \) after 13 iterations with \( \epsilon_a = 0.04\% \). Therefore,

\[
x = 0 + 3.5(0.27893) = 0.976257 \\
y = 0 + 2(0.27893) = 0.557861
\]

14.8

\[
\frac{\partial f}{\partial x} = -8 + 2x - 2y \\
\frac{\partial f}{\partial y} = 12 + 8y - 2x
\]

At \( x = y = 0 \),

\[
\frac{\partial f}{\partial x} = -8 \\
\frac{\partial f}{\partial y} = 12
\]

\[
f(0 - 8h, 0 + 12h) = g(h) \\
g(h) = 832h^2 + 208h
\]

At \( g'(h) = 0 \), \( h^* = -0.125 \).

Therefore,

\[
x = 0 - 8(-0.125) = 1 \\
y = 0 + 12(-0.125) = -1.5
\]