EAD 115

Numerical Solution of Engineering and Scientific Problems

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Interpolation

• Given a set of points \((x_i, y_i)\), an interpolating function is one which is defined for all \(x\) in the range of the \(x_i\), and which satisfies \(f(x_i) = y_i\).
• Polynomials are a convenient class of functions to use for this purpose, though others such as splines are also used.
• There are different ways to express the same polynomial.
• Given \(n\) points, we can in general determine an \(n-1\) degree polynomial that interpolates them.
Linear function
Two points
Degree one

Quadratic function
Three points
Degree Two

Cubic function
Four points
Degree three
Linear Interpolation
\[
\frac{f_1(x) - f(x_0)}{x-x_0} = \frac{f(x_1) - f(x_0)}{x_1-x_0}
\]

\[
f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1-x_0}(x-x_0)
\]

\[
\ln(1) = 0
\]
\[
\ln(4) = 1.386294
\]

\[
\ln(x) \overset{\text{def}}{=} f_1(x) = 0 + \frac{1.386294 - 0}{4-1}(x-1)
\]

\[
\ln(2) \overset{\text{def}}{=} f_1(2) = \frac{1.386294}{3}(2-1) = 0.4620981
\]

\[
\ln(2) = 0.6931472
\]
The diagram illustrates the function $f(x) = \ln x$. The black curve represents the true value of $f(x)$, while the blue lines are linear estimates. The graph shows the function and its estimates over the range of $x$ from 0 to 5.
Quadratic Interpolation

• Three points determine a quadratic
• This should fit many functions better than linear interpolation
• We derive a general form for quadratic interpolation
• We then derive a method to estimate the three unknowns (coefficients) that determine a quadratic function
The graph illustrates the function $f(x) = \ln x$ and its approximations. The true value is shown as a solid line, with two approximations: a quadratic estimate and a linear estimate. The quadratic estimate is represented by a dashed line that closely follows the true value, while the linear estimate is a straight line that deviates more as $x$ increases.
\[ f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) \]
\[ = b_0 + b_1x - b_1x_0 + b_2x^2 + b_2x_0x_1 - b_2xx_0 - b_2xx_1 \]
which is of the form
\[ = a_0 + a_1x + a_2x^2 \]
with
\[ a_0 = b_0 - b_1x_0 + b_2x_0x_1 \]
\[ a_1 = b_1 - b_2x_0 - b_2x_1 \]
\[ a_2 = b_2 \]
which shows either form is general
\[ f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) \]
\[ f(x_0) = b_0 \]
\[ b_0 = f(x_0) \]
\[ f(x_1) = b_0 + b_1(x_1 - x_0) \]
\[ f(x_1) = f(x_0) + b_1(x_1 - x_0) \]
\[ b_1 = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} \]
\[ f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) \]
\[ b_0 = f(x_0) \]
\[ b_1 = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} \]
\[ f(x_2) = f(x_0) + \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1) \]
\[ \frac{f(x_2) - f(x_0)}{(x_2 - x_0)(x_2 - x_1)} = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} - \frac{(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} + b_2 \]
\[ b_2 = \frac{f(x_2) - f(x_0)}{(x_2 - x_0)(x_2 - x_1)} - \frac{f(x_1) - f(x_0)}{(x_1 - x_0)(x_2 - x_1)} \]
\[ b_2 = \frac{f(x_2) - f(x_0)}{(x_2 - x_0)(x_2 - x_1)} - \frac{f(x_1) - f(x_0)}{(x_1 - x_0)(x_2 - x_1)} \]

\[ b_2 = \frac{f(x_2) - f(x_1)}{(x_2 - x_0)(x_2 - x_1)} + \frac{f(x_1) - f(x_0)}{(x_1 - x_0)(x_2 - x_1)} - \frac{f(x_1) - f(x_0)}{(x_1 - x_0)(x_2 - x_1)} \]

\[ b_2 = \frac{f(x_2) - f(x_1)}{(x_2 - x_0)(x_2 - x_1)} + \left[ \frac{f(x_1) - f(x_0)}{(x_2 - x_0)(x_2 - x_1)(x_1 - x_0)} \right] \]

\[ b_2 = \frac{f(x_2) - f(x_1)}{(x_2 - x_0)(x_2 - x_1)} - \frac{f(x_1) - f(x_0)}{(x_2 - x_0)(x_1 - x_0)} \]

\[ b_2 = \frac{f(x_2) - f(x_1)}{(x_2 - x_0)} - \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} \]
\[ f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) \]

\[ b_0 = f(x_0) \]

\[ b_1 = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} \]

looks like a finite first divided difference

\[ f(x_2) - f(x_1) = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} \]

\[ b_2 = \frac{(x_2 - x_1)(x_1 - x_0)}{(x_2 - x_0)} \]

looks like a finite second divided difference
Approximate \( \ln(2) = 0.6931472 \) by interpolating

\( (1, 0) \)

\( (4, 1.386294) \)

\( (6, 1.791759) \)

\( b_0 = f(x_0) = 0 \)

\[
b_1 = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} = \frac{1.386294 - 0}{3} = 0.4620981
\]

\[
b_2 = \left( \frac{f(x_2) - f(x_1)}{(x_2 - x_1)} \right) \left( \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} \right) = \frac{1.791759 - 1.386294}{2} \cdot \frac{-0.4620981}{5} = -0.0518731
\]

\[
f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)
\]

\[
f_2(2) = 0 + 0.4620981(2 - 0) - 0.0518731(2 - 0)(2 - 4) = 0.5658444
\]
General Form of Newton’s Divided Difference Interpolating Polynomials

• The order $n$ polynomial interpolates $n+1$ points
• The coefficients are finite divided differences
• They can be calculated recursively
\[ f_n(x) = b_0 + b_1(x - x_0) + b_1(x - x_0)(x - x_1) + \cdots + b_n(x - x_0)(x - x_1) \cdots (x - x_n) \]

\[ b_0 = f(x_0) \]

\[ b_1 = f[x_1, x_0] \]

\[ b_2 = f[x_2, x_1, x_0] \]

\[ \cdots \]

\[ b_n = f[x_n, x_{n-1}, \ldots, x_1, x_0] \]

\[ f[x_i, x_{i-1}, \ldots, x_1, x_0] = \frac{f[x_i, x_{i-1}, \ldots, x_2, x_1] - f[x_{i-1}, x_{i-2}, \ldots, x_1, x_0]}{x_i - x_0} \]
<table>
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<tr>
<th>$i$</th>
<th>$x_i$</th>
<th>$f(x_i)$</th>
<th>First</th>
<th>Second</th>
<th>Third</th>
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<tbody>
<tr>
<td>0</td>
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<td>$f(x_0)$</td>
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<td>$f[x_2, x_1, x_0]$</td>
<td>$f[x_3, x_2, x_1, x_0]$</td>
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<td>$f(x_1)$</td>
<td>$f[x_2, x_1]$</td>
<td>$f[x_3, x_2, x_1]$</td>
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<tr>
<td>2</td>
<td>$x_2$</td>
<td>$f(x_2)$</td>
<td></td>
<td>$f[x_3, x_2]$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$x_3$</td>
<td>$f(x_3)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>2nd dd</td>
<td>3rd dd</td>
<td>4th dd</td>
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<td>-0.20007</td>
<td>0.051307</td>
<td>-0.01095</td>
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<tr>
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<td>-0.10003</td>
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</table>

|                   |          |          |          |          | 0.693147 |
|                   |          |          |          |          | -0.00218 |
Lagrange Interpolating Polynomial

• Given \( n+1 \) points and function values, there is only one degree-\( n \) polynomial going through the points.

• The Lagrange formulation is thus equivalent, leading to the same interpolating polynomial.

• It is easier to calculate.
\[ f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i) \]

\[ L_i(x) = \prod_{j=0}^{n} \frac{x - x_j}{x_i - x_j} \]

This passes through each of the points because when \( x = x_k \), all of the \( L_i(x) \) are 0 except for \( L_k(x) \), which is equal to 1.